Properties of transformations

The fundamental questions of existence and uniqueness can be expressed using transformations, and these properties are given names which should learn, even if these non-standard uses of English words make your eyes glaze over.

Math really can be a foreign language, and it should be studied as one. Each word we define has a precise meaning, independent of any other definitions of the word you may already know. You are expected to know the definitions. Do not pretend to yourself that you don't.

Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. If every $\vec{b} \in \mathbb{R}^m$ is the image of some $\vec{x} \in \mathbb{R}^n$ under T, then T is called onto (or surjective).
   Equiv: $T(\mathbb{R}^n) = \mathbb{R}^m$.

   For a matrix trans., this is saying $A\vec{x} = \vec{b}$ is solvable no matter the $\vec{b}$. So,

   * $T(\vec{x}) = A\vec{x}$ is onto $\iff$ A has a pivot in every row.

2. If $T(\vec{x}) = T(\vec{y})$ implies $\vec{x} = \vec{y}$, then T is called one-to-one (or injective)
   Equiv: $T(\vec{x}) = \vec{b}$ has at most one solution $\vec{x}$ for each $\vec{b}$ (but maybe no solutions)
   Equiv: Each $\vec{b} \in \mathbb{R}^m$ is the image of at most one $\vec{x} \in \mathbb{R}^n$.

   * $T(\vec{x}) = A\vec{x}$ one-to-one $\iff$ A has a pivot in every column.

You will be tempted to not learn these words. This is not calculus or high school math where you have a handful of words to remember, and you have plenty of time for osmosis to occur. Just make flash cards with the definitions (and with facts like the relationship to pivots, above)!

Onto: the entire codomain is covered by the image of the domain.
(≥ 1 solution)

ex $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ (projection of $\mathbb{R}^3$ onto xy plane – for instance, $\begin{bmatrix} x \\ y \end{bmatrix}$ is an image of $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$)

One-to-one: ("horizontal line test" from (pre)calculus, though this makes less sense for higher dimensions!)
(≤ 1 solution)
can undo the action of T, since for $\vec{x} \in \mathbb{R}^n$, take $\vec{b} = T(\vec{x})$. Solve $\vec{b} = T(\vec{y})$ for $\vec{y}$. The only solution is $\vec{y} = \vec{x}$!

ex $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Not always solvable but unique when it is. (inclusion of $\mathbb{R}^2$ in $\mathbb{R}^3$ as xy)

For a linear transformation, the test for one-to-one is somewhat simpler than expected. Solutions to $T(\vec{x}) = \vec{0}$, like for homogeneous systems, correspond to solutions to $T(\vec{x}) = \vec{b}$ when there is at least one particular solution.

Recall: $T(\vec{x}_p + \vec{x}_h) = T(\vec{x}_p) + T(\vec{x}_h) = \vec{b} + \vec{0} = \vec{b}$,
so $\vec{x}_p + \vec{x}_h$ is a part. solution

and $T(\vec{x}_p - \vec{x}_q) = T(\vec{x}_p) - T(\vec{x}_q) = \vec{b} - \vec{b} = \vec{0}$
so $\vec{x}_p - \vec{x}_q$ is a homog. sol.

Thus, T onto $\Longleftrightarrow$ $T(\vec{x}) = \vec{0}$ has only trivial sol.

For matrix transformations, this is $\Leftrightarrow A\vec{x}=\vec{0}$ having only the triv. sol. $\Leftrightarrow \text{null}(A) = \{\vec{0}\}$.

moral: nullspace of $[T]$ controls whether one-to-one.
Recall: this is "standard matrix".

ex $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$. The columns are dependent. Thus, $\vec{x} \mapsto A\vec{x}$ is not one-to-one.

It is also not onto, since only one row has a pivot.

Matrix operations

Let A be $m\times n$ with columns $\vec{a}_1, \ldots, \vec{a}_n$. We have been writing A as $[\vec{a}_1 \cdots \vec{a}_n]$. The element in row i and column j, element (i,j), is denoted a_{ij} when $A = [a_{ij}]_{ij}$. That is, when

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}.$$

ex A diagonal matrix is a square matrix ($n=m$) with $a_{ij}=0$ whenever $i \neq j$. An identity matrix is one with $a_{ii}=1$.

Just like vectors, matrices have addition and scalar multiplication defined element-wise.

ex $2\begin{bmatrix} 1 & -3 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ 8 & 11 \end{bmatrix}$

All the same algebraic rules apply.

But, unlike vectors, there is also a multiplication between matrices of compatible sizes.

The motivation: A is $m\times n$, B is $n\times p$, $\vec{x}\in\mathbb{R}^p$.

If we compute $A(B\vec{x})$, $B\vec{x}\in\mathbb{R}^n$, so $A(B\vec{x})\in\mathbb{R}^m$.
Can we define an AB so that

$$A(B\vec{x}) = (AB)\vec{x} \ ?$$

The answer is yes. We can think of a transformation $T:\mathbb{R}^p\to\mathbb{R}^m$ defined by $\vec{x}\mapsto A(B\vec{x})$.
Is it linear?

(i) $T(\vec{u}+\vec{v}) = A(B(\vec{u}+\vec{v})) = A(B\vec{u}+B\vec{v}) = A(B\vec{u})+A(B\vec{v})$
$= T(\vec{u}) + T(\vec{v})$

(ii) $T(c\vec{u}) = A(B(c\vec{u})) = A(cB\vec{u}) = cA(B(\vec{u})) = cT(\vec{u})$.

Yes it is. This means it has a matrix. We have

$$A(B(\vec{x})) = T(\vec{x}) = \underbrace{[T]}_{\text{this is whatever } AB \text{ is.}}\vec{x}$$

This is not difficult to calculate. Remember:

$$[T] = [T(\vec{e}_1)\ T(\vec{e}_2)\ \cdots\ T(\vec{e}_p)] \quad \text{where}$$

$$\vec{e}_1 = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} \cdots \quad \text{(the columns of the identity matrix)}$$

example: $T(\vec{e}_1) = A(B(\vec{e}_1)) = A\vec{b}_1$
$\vdots$
$T(\vec{e}_i) = A(B(\vec{e}_i)) = A\vec{b}_i$

So AB is equal to $[A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_p]$.

ex $A=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ is rotation 90° ccw

$B=\begin{bmatrix}0&1\\-1&0\end{bmatrix}$ is rotation 90° cw

$$AB=\begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\-1&0\end{bmatrix}=\left[0\begin{bmatrix}0\\1\end{bmatrix}-\begin{bmatrix}-1\\0\end{bmatrix}\quad 1\begin{bmatrix}0\\1\end{bmatrix}+0\begin{bmatrix}-1\\0\end{bmatrix}\right]$$
$$=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$$

In practice, we use the quick way to calculate $A\vec{b}_i$ directly rather than as a lin. comb.

ex $$\begin{bmatrix}1&c\\0&1\end{bmatrix}\begin{bmatrix}1&0\\d&1\end{bmatrix}=\begin{bmatrix}1\cdot1+c\cdot d & 1\cdot0+c\cdot1\\0\cdot1+1\cdot d & 0\cdot0+1\cdot1\end{bmatrix}=\begin{bmatrix}1+cd&c\\1+d&1\end{bmatrix}$$

motto: element (i,j) of AB is row i of A and column j of B.

Here is a way to organize this I never use:

ex $$\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}7&9\\8&10\end{bmatrix}$$

$$\begin{array}{cc} & \begin{bmatrix}7&9\\8&10\end{bmatrix}\\ \begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix} & \begin{bmatrix}1\cdot7+4\cdot8 & 1\cdot9+4\cdot10\\2\cdot7+5\cdot8 & 2\cdot9+5\cdot10\\3\cdot7+6\cdot8 & 3\cdot9+6\cdot10\end{bmatrix}\end{array}$$

3×2 times $2\times2 \Rightarrow$ result 3×2. In general, A $m\times n$, B $n\times p$, AB is $m\times p$ (n's agree, A's rows, B's columns).

ex Can we multiply 2×3 by 4×2?
No: $3 \neq 2$.

It is possible to compute the (i,j) element directly. Let $C = AB$. Then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Aside: with Einstein summation notation, we write vectors as x_i, matrices as a^i_j. $y_j = a^i_j x_i$ is matrix-vector product. For b^i_j also a matrix, $c^j_i = a^k_i b^j_k$ is AB. If you want to know more, come to office hours or ask on Piazza.

Example properties: $A(BC) = (AB)C$

$A(B+C) = AB + AC$

$I_n A = A I_n = A$, $\quad I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

Warning: $AB \neq BA$ in general!

ex $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ not equal

ex $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ rot by 90°CCW then proj onto x-axis

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ proj onto x-axis then rot by 90°CCW.

Warning: $AB = 0 \not\Rightarrow A = 0$ or $B = 0$

ex $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Warning: If $AB = AC \not\Rightarrow B = C$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 22 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

not equal! (has to do with pivots of A — need one in each column)

Transpose of A, denoted A^T, is the matrix flipped over major diagonal. (i,j) entry is a_{ji}.

ex $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Rules:
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(rA)^T = rA^T$ $(r \in \mathbb{R})$
- $(AB)^T = B^T A^T$ ← reversed!

Proof of last rule:

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k B_{ki} A_{jk} = \sum (B^T)_{ik} (A^T)_{kj}$$

$$= (B^T A^T)_{ij}.$$