Vectors in $\mathbb{R}^n$

A solution to a linear system is an ordered list of numbers. It turns out to be meaningful to perform operations such as addition on solutions, so we will define "vectors" to hold onto such lists. In fact, we can simply define a vector in $\mathbb{R}^n$ to be an $n \times 1$ matrix. These are also called column vectors or n-dimensional Euclidean vectors.

Note We will later generalize the definition of vectors from this one. I am being careful in saying these are vectors in $\mathbb{R}^n$ for this reason. Future vectors need not have entries, even.

We use arrow hats or a bold face to denote vector variables (though advanced texts will often omit either).

Examples These are vectors in $\mathbb{R}^2$:

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

but not $\begin{bmatrix} 1 & 2 \end{bmatrix}$ ("row vector") nor $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$. ("in $\mathbb{R}^3$")

Sometimes we are sloppy and have $(1,2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, especially when space is a premium.

Two vectors are equal if they have the same entries, in the same order. Otherwise, they are not equal.

Vector sum/addition and vector difference are defined entrywise (componentwise).

example
$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}1+2\\2+3\end{bmatrix}=\begin{bmatrix}3\\5\end{bmatrix}$$
$$\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}-2\\2\end{bmatrix}=\begin{bmatrix}1-(-2)\\2-2\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}$$

Vectors also have an operation of scalar multiplication, named for the geometric action of scaling. This is a multiplication between a real number (called the scalar) and a vector. $c\vec{v}$ is the scalar multiple of $\vec{v}$ by c.

example
$$3\begin{bmatrix}4\\-2\end{bmatrix}=\begin{bmatrix}3\cdot 4\\3(-2)\end{bmatrix}=\begin{bmatrix}12\\-6\end{bmatrix}$$
$$-\begin{bmatrix}4\\-2\end{bmatrix}=\begin{bmatrix}-1\cdot 4\\-1(-2)\end{bmatrix}=\begin{bmatrix}-4\\2\end{bmatrix}$$

Notice that vectors are number-like, except multiplication has a scalar as one operand, and there is no division except for $\vec{v}/c=\frac{1}{c}\vec{v}$ (when $c\neq 0$). In fact, they satisfy many familiar algebraic properties:

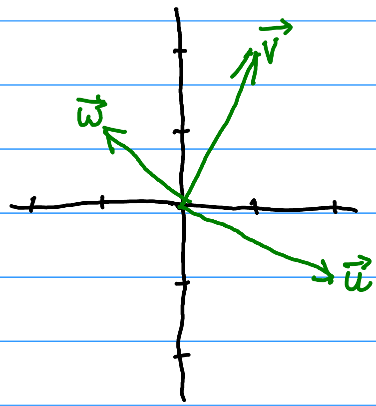
for $\vec{u},\vec{v},\vec{w}\in\mathbb{R}^n$, $c,d\in\mathbb{R}$, $\vec{0}=\begin{bmatrix}0\\\vdots\\0\end{bmatrix}\in\mathbb{R}^n$,

- $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
- $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
- $\vec{u}+\vec{0}=\vec{u}$
- $\vec{u}+(-\vec{u})=\vec{0}$
- $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$
- $(c+d)\vec{u}=c\vec{u}+d\vec{u}$
- $c(d\vec{u})=(cd)\vec{u}$
- $1\vec{u}=\vec{u}$

These are all true simply because the arithmetic is defined entrywise, so the corresponding rules for $\mathbb{R}$ carry over.

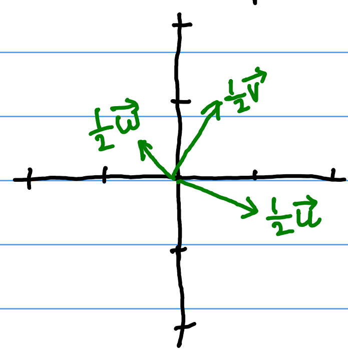
The geometric picture for $\mathbb{R}^2$ (and, on n-dimensional paper, $\mathbb{R}^n$) is arrows on the plane, starting at the origin.

example $\quad \vec{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Scalar multiplication corresponds to scaling the whole plane.

example

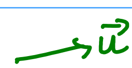
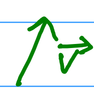
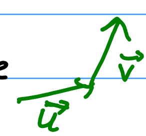
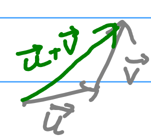


This is $\frac{1}{2}$ of each vector from the previous example.

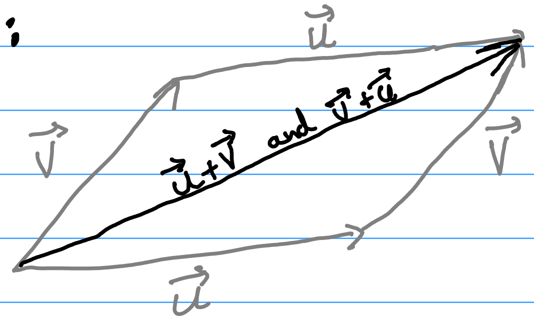
To understand vector addition, we will allow the arrows to translate (shift) across the plane, all the while being considered to be the same vector.

In other words, a vector is a direction and a magnitude.

Then, vector addition proceeds by the tip-to-tail principle:

To compute $\vec{u} + \vec{v}$, for  and  , move $\vec{v}$'s tail to $\vec{u}$'s tip, like  , and then the resulting sum is  , the vector from the first tail to the last tip.

The fact $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ is the following parallelogram:



Linear combinations

Suppose $\vec{v}_1, \cdots, \vec{v}_k \in \mathbb{R}^n$ are vectors. If you take a look at the algebraic properties for vectors, thinking "what is everything I could possibly do using addition and scalar multiplication to combine these vectors," you would eventually realize everything possible can be simplified into the following form:

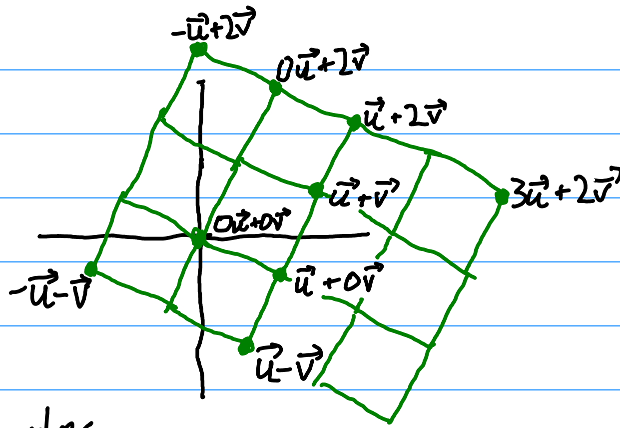
$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

for some scalars $c_1, \cdots, c_k \in \mathbb{R}$. This is called a linear combination of $\vec{v}_1, \cdots, \vec{v}_k$ with coefficients or weights $c_1, \cdots, c_k$.

examples for $\vec{u}=\begin{bmatrix}2\\-1\end{bmatrix}$, $\vec{v}=\begin{bmatrix}1\\2\end{bmatrix}$

- $3\vec{u}+2\vec{v}=\begin{bmatrix}3(2)+2(1)\\3(-1)+2(2)\end{bmatrix}=\begin{bmatrix}8\\1\end{bmatrix}$
- $0\vec{u}+0\vec{v}=\begin{bmatrix}0(2)+0(1)\\0(-1)+0(2)\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$
- $\vec{u}-\vec{v}=\begin{bmatrix}2-1\\-1-2\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}$

Graphically,



Notice coefficients are like coordinates on the grid!

Vector equations

A system of equations $\left[\begin{array}{cccc|c} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$

with $\vec{a}_1, \dots, \vec{a}_n, \vec{b} \in \mathbb{R}^m$ can be equivalently written as

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}.$$

That is, the question of finding solutions to a system of equations is equivalent to the question of whether $\vec{b}$ is a linear combination of $\vec{a}_1, \dots, \vec{a}_n$. The corresponding collection of weights forms a solution to the system.

It is useful to define the set of all $\vec{b}$ such that the system has a solution:

Definition For $\vec{a}_1, \cdots, \vec{a}_n \in \mathbb{R}^m$, $\operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$ is the set of all linear combinations of $\vec{a}_1, \cdots, \vec{a}_n$. (in symbols, it is $\{c_1\vec{a}_1 + \cdots + c_n\vec{a}_n \mid c_1, \cdots, c_n \in \mathbb{R}\}$.

Thus, a system $[\vec{a}_1 \cdots \vec{a}_n | \vec{b}]$ is consistent if $\vec{b} \in \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\}$, and vice versa.

example $\begin{cases} 2x_1 + 4x_2 = g \\ x_1 + 2x_2 = h \end{cases}$ has solutions for which g, h?

same as $\begin{bmatrix} 2x_1 + 4x_2 \\ x_1 + 2x_2 \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$

same as $\begin{bmatrix} 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} 4x_2 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$

same as $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} g \\ h \end{bmatrix}$

(as predicted)

So the question is equivalently,

when is $\begin{bmatrix} g \\ h \end{bmatrix} \in \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\}$.

We don't have tools yet to do this in general, but notice here that

$$x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= (x_1 + 2x_2) \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So $\text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\}$.

Can conclude $g = 2h$ necessary and sufficient.

Another notation for a linear combination is the matrix-vector product $A\vec{x}$ of a $m \times n$ matrix A and $\vec{x} \in \mathbb{R}^n$ defined by

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

where $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$

and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

For a system $\left[\begin{array}{ccc|c} \vec{a}_1 & \cdots & \vec{a}_n & \vec{b} \end{array}\right]$, with A the coefficient matrix, a solution is a vector $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

example $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 1 & 5 \end{array}\right]$ or $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Since $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(1) + 1(2) \\ 2(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$,

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a solution to the system.

In the new notation, $\operatorname{Span}\{\vec{a}_1, \cdots, \vec{a}_n\} = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$, where $A = [\vec{a}_1 \ \cdots \ \vec{a}_n]$.

Theorem Let A be $m \times n$. The following are equivalent.

a. For each $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solution.

b. Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$

$$(\mathbb{R}^m = \text{Span}\{\vec{a}_1, \cdots, \vec{a}_n\})$$

d. A has a pivot in every row.

(and (a) is equivalent to $[A \mid \vec{b}]$ being a consistent system for each $\vec{b}$)