1.1

A matrix is a rectangular array of numbers, which is used to hold onto such data as an aggregate object.

Examples $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$ $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ $\begin{bmatrix} 22 \end{bmatrix}$ $\begin{bmatrix} 0 & 3 & 7 \end{bmatrix}$

or $\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$ $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \end{pmatrix}$ $\begin{pmatrix} 0 & 3 & 7 \end{pmatrix}$

The choice of brackets vs. parentheses is immaterial.

History 1848 — Sylvester came up with matrices as a terse notation for linear systems. "matrix" is Latin for "womb" (which makes the movie's title make more sense) holds numbers in place, generates determinants

1858 — Caley uses letters in place of matrices and figures out an "algebra" (matrix addition and multiplication)

Consider the system

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\quad 2x_2 - 8x_3 = 8 \\ -4x_1 + 5x_2 + 9x_3 = -9 \end{cases}$$

There are two matrices we may form from this:

1) The coefficient matrix or matrix of coefficients

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

which is principally used when we wish to solve multiple systems which are the same except for their constant terms (and, more generally, matrix algebra and linear transformations, later)

2) the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right]$$

This dotted line is optional, but will prevent confusion. It represents that the coefficient matrix has been augmented with a column of constants

A matrix with n rows and m columns is said to have size $n \times m$ or be an $n \times m$ matrix.

pronounced "n by m"

Rows first!

$$\begin{array}{c} \\ \text{2 rows} \end{array} \overset{\text{3 columns}}{\begin{bmatrix} 2 & 2 & 6 \\ 2 & -2 & 2 \end{bmatrix}}$$

Ex The preceding matrix, thought of as an augmented matrix, corresponds to the system

$$\begin{cases} 2x_1 + 2x_2 = 6 \\ 2x_1 - 2x_2 = 2 \end{cases} \qquad (2 \text{ var}, \ 2 \text{ eqn.})$$

Or, as a coefficient matrix, could have come from

$$\begin{cases} 2x_1 + 2x_2 + 6x_3 = 7 \\ 2x_1 - 2x_2 + 2x_3 = 19 \end{cases} \qquad (3 \text{ var}, \ 2 \text{ eqn.})$$

(arbitrary)

Solving linear systems

We want to find rules which we may use to transform a system into a simpler, equivalent system.

Recall: systems are equivalent if they have same solution sets

As it turns out, we only need three rules, called the elementary row operations. We will see every other possible valid rule is just a combination of these.

Rule: Interchange or swap. Interchanging two equations does not change the solution set.

ex $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 3 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \end{array}\right]$

Rule: Scaling. Multiplying an equation by a nonzero constant (a scalar) does not change the solution set.

ex $\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 2 & 4 \end{array}\right] \xrightarrow{\frac{1}{2}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \end{array}\right]$

Rule: Replacement. Adding a scale multiple of one row to another does not change the solution set.

ex $\left[\begin{array}{cc|c} 1 & 3 & 3 \\ 0 & 1 & 2 \end{array}\right] \xrightarrow{R_1 - 3R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 2 \end{array}\right]$

This rule in theory could instead be "add a row to another", since $R_1 + kR_2 \to R_1$ could be the sequence

$$kR_2 \to R_2$$
$$R_1 + R_2 \to R_1 \qquad \text{(so long as } k \neq 0\text{)}$$
$$\frac{1}{k}R_2 \to R_2$$

but, pragmatically, that is more work.

Two matrices are row equivalent if there is a sequence of elementary row operations transforming the first into the second. $A \sim B$ sometimes denotes this relation. Note reversible: $B \sim A$ if $A \sim B$.

If two augmented matrices are row equivalent, then their associated systems are equivalent.

(The converse is not true:

$$\begin{cases} x_1 = 1 \\ x_1 = 2 \end{cases} \quad \text{and} \quad \begin{cases} x_2 = 1 \\ x_2 = 2 \end{cases} \quad \text{equivalent (2 var, 2 equ)}$$

but $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 1 & 0 & 2 \end{array}\right]$ and $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ are not row equivalent. This is somewhat pathological.)

Ex
$$\begin{cases} x_1 - 5x_2 = 0 \\ 2x_1 - 7x_2 = 3 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & -5 & 0 \\ 2 & -7 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1 \to R_2} \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{\frac{1}{3}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1 + 5R_2 \to R_1} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

So solution set is $\{(5,1)\}$, as this corresponds to the system $\begin{cases} x_1 = 5 \\ x_2 = 1 \end{cases}$.

The technique is "elimination", or "row reduction."

1.2 Row reduction

The goal is to use row operations to put a matrix into (row) echelon form. These are the properties to identify a matrix in this form:

1. all all-zero rows are the last rows

yes $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ no $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

2. each leading entry of a row is to the right of leading entries of preceding rows (where a leading entry is the first non-zero entry of a row)

yes $\begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 7 \end{pmatrix}$ $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 7 \end{pmatrix}$

no $\begin{pmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

3. entries below a leading entry are all 0.

no $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 4 \end{pmatrix}$ (also violates 2)

A reduced (row) echelon form matrix also satisfies

4. leading entries are 1

yes $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ no $\begin{pmatrix} 2 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$

5. the other entries in a column with a leading entry are all 0.

yes $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$ no $\begin{pmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \end{pmatrix}$

Theorem A matrix A is row equivalent to exactly one reduced row echelon form matrix.
Call it rref(A).

A pivot of A is the position of a leading entry in rref(A), and a pivot column is a column of A having a pivot.

Strategy for computing rref(A):

- for each non-zero column,
  1. swap a row if necessary so the pivot position in this column is nonzero
  2. scale the row so the pivot is 1
  3. use replacement to eliminate all nonzero entries below pivot
- now matrix is in row echelon form
- begin "back substitution"
  for each pivot column, right-to-left:
  eliminate non-zero entries above pivot (by replacement)

ex

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 4 & 1 \\ 1 & 3 & 9 & 3 \end{array}\right] \quad \begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 8 & 3 \end{array}\right]$$

$$R_3 - 2R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \quad \tfrac{1}{2}R_3 \to R_3 \quad \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]}_{\text{(ref)}}$$

$$\begin{array}{c} R_1 - R_3 \to R_1 \\ R_2 - 3R_3 \to R_2 \end{array} \quad \begin{bmatrix} 1 & 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \quad R_1 - 1R_2 \to R_2 \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}}_{\text{(rref)}}$$

as a system, get $(x_1, x_2, x_3) = (0, \frac{-1}{2}, \frac{1}{2})$ (should check)

Please write steps explicitly for validation purposes!

ex $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 0 & 1 & 6 \end{array}\right]$ is in ref.

$\xrightarrow{R_1 - R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 6 \end{array}\right]$ is in rref.

so columns 1,3 are pivot columns.

A parametric solution gives the solution to a system in terms of the non-pivot-columns' variables.

For above, $\begin{cases} x_1 + 2x_2 = -2 \\ x_3 = 6 \end{cases}$, so

$\begin{cases} x_1 = -2 - 2x_2 \\ x_2 \text{ is free} \\ x_3 = 6 \end{cases}$ is a parametric solution.

This is relatively easy to read off rref(A).

inconsistent = no solutions

ex $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right]$ since $0=1$ can't happen.
(rule: check if augmented column is pivot column)

else consistent = solutions exist

in this case, either unique solutions or non-unique solutions. Check if there are free (non-pivot) columns: if there are any, the system has at most one solution.

ex $\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$ consistent and non-unique

$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$ consistent and unique

<u>ex</u> $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 7 \end{array}\right]$ $\begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array}$ $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{array}\right]$

can already see this is a pivot column so inconsistent.

Note also $0 \neq -2$!

"overconstrained". too many, conflicting equations.

$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{array}\right]$ consistent.