Monday, June 20.

Chapter 1.1. Linear systems

A variable* is a formal symbol representing indeterminacy. By the process of substitution a variable may stand for a value. But a variable is not in itself a value, strictly speaking. We will speak loosely and say things like "$x = 3$" to mean "substitute 3 for x."

*or "unknown"

In addition to x, y, z, w, we need an endless supply of variables in linear algebra, so we also introduce the likes of $x_1, \dots, x_n, \dots$.

Do not confuse constants with variables. The distinction is made through context. For instance, $y = ax$ has "a" a constant because I declare it to be so. While it is a variable in the sense it can stand for different values, it is constant in the sense we assume the substitution has already been done.

definition A linear equation is an equation of the form $c_1 x_1 + \cdots + c_n x_n = b$, with $c_1, \dots, c_n, b$ constants and $x_1, \dots, x_n$ variables

That's just the formal definition. It is a bit too rigid for actual use, so we also (informally) allow any equation which is algebraically equivalent to a linear equation to itself be called "linear."

Examples

$$2x = 1$$
$$3x + 4y = 7$$
$$x_1 + 2x_2 + 3x_3 = 4$$
$$y = 2x + 1$$
$$x_2 = 2(\sqrt{6} - x_1) + x_3$$

Non-examples

$$4x_1 - 5x_2 = x_1 x_2$$
$$1 = xy$$
$$x_2 = \sqrt{x_1}$$

Definition A <u>system of linear equations</u> (or "linear system" or "system of equations" or "system") is a collection of linear equations. Order matters.

ie., swapping the order of equations gives a different system

ex

$$\begin{cases} y = 2x + 3 \\ x = 1 \end{cases}$$

Definition A <u>solution</u> to a system is a substitution $(s_1, \cdots, s_n)$ of values for variables $(x_1, \cdots, x_n)$ which simultaneously satisfies each equation.

ex For $\begin{cases} y = 2x+3 \\ x = 1, \end{cases}$ $(1,5)$ is a solution (assuming variable order (x,y)).

Aside: Describing a system

Whenever you have a system, it must always be clear the number of unknowns. For instance, in the above example, maybe there were three unknowns, and then $(x,y,z) = (1,5,22)$ is also a solution.

To be explicit, we describe the system as "two equations and two variables," or "two equations in/with two unknowns" or some combination.

definition A solution set for a system is a set of solutions for a system.

Note how a solution set is something a system ends up having, and is not part of its definition. We are paving the way toward formal manipulation of equations without any regard toward whatever the solutions may be. Variables as formal symbols is also important toward this end.

Examples (assume variable order x, y)

1) $\begin{cases} y = 2x+3 \\ x = 1 \end{cases}$ has s.set $\{(1,5)\}$

2) $\begin{cases} y = 2x+3 \\ y = 2x+4 \end{cases}$ has s.set $\{\}$ or $\emptyset$ (the "empty set")

3) $\{ y = 2x+3$ has s.set $\{(x, 2x+3) \mid x \in \mathbb{R}\}$

also written $\{(x, 2x+3) : x \in \mathbb{R}\}$

"the set of all $(x, 2x+3)$ where x ranges over all real numbers"

4) $\{$ (no equations) has s.set $\{(x,y) \mid x \in \mathbb{R}$ and $y \in \mathbb{R}\}$

that is, every substitution is a solution!

Definition Two systems are equivalent if they have the same solution set.

This is giving a name to an important concept we will be elaborating, which happens to work out for linear systems:

To solve a system, sequentially replace it with "simpler" equivalent systems until it is obviously solved.

It is remarkable that we can do this at all (Gaussian elimination, soon)!

Silly example $\begin{cases} y = 2x+3 \\ x=1 \end{cases}$ is equivalent

to $\begin{cases} x=1 \\ y=2x+3 \end{cases}$

Less silly it is equivalent to $\begin{cases} x=1 \\ y=5 \end{cases}$

(*) Ex $\begin{cases} x+y = 2 \\ x-y = 6 \end{cases}$ is equivalent to

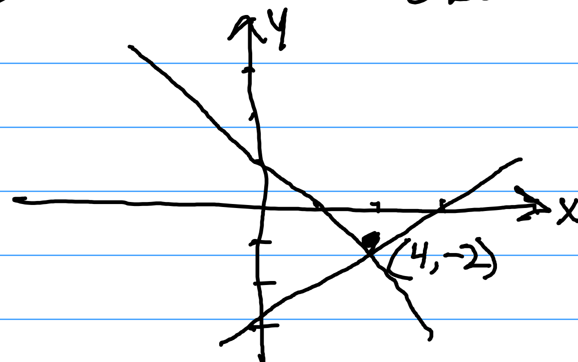
$\begin{cases} x+y = 2 \\ 2x = 8 \end{cases}$ (adding 1^{st} eqn to 2^{nd})

is equivalent to $\begin{cases} x+y = 2 \\ x = 4 \end{cases}$

is equivalent to $\begin{cases} y=-2 \\ x=4 \end{cases}$.

With 2 variables, we can understand a linear equation as a line. For the above example (*), we can draw

(using x-, y-intercepts)



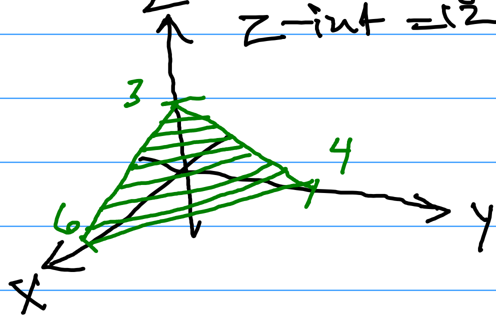
In 3D With 3 variables, x, y, z, we can draw pictures of the solution set to a linear system (with some difficulty).

ex $2x + 3y + 4z = 12$ has the following intercepts:

$x\text{-int} = \frac{12}{2} = 6$

$y\text{-int} = \frac{12}{3} = 4$

$z\text{-int} = 12/4 = 3.$

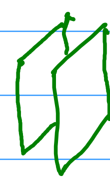


I have only drawn its intersection with the 1st octant, for clarity.

Notice the equation describes a plane.

What are ways planes may intersect?

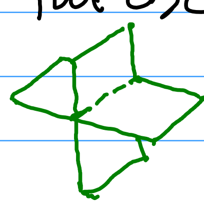
I) They might be parallel.



(no intersection).

II) They might be the same plane (intersection is the plane itself)

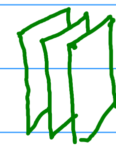
III) They might intersect at a line



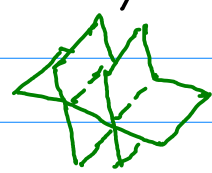
Three planes gets more complicated.

I) At least two are the same: see the two plane case.

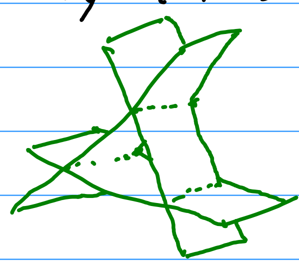
II) All three are parallel (no intersection)



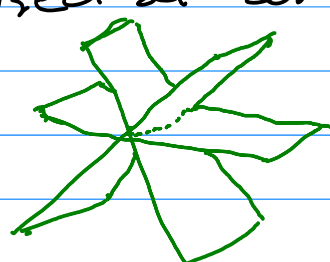
III) Two parallel, third isn't (it intersects first two at a pair of parallel lines. No intersection.)



IV) Mutually intersect at three parallel lines (no intersection)



V) Intersect at common line (line of intersection)



VI) Intersect at a single point