MATH 54 LECTURE 3. FINAL EXAM
AUGUST 12, 2016, 110 MINUTES
(10 PAGES)

Problem Number	1	2	3	4	5	6	7	Total
Score								

YOUR NAME: ______________________________

No calculators, no cell phones, no references except for one 8.5×11 *sheet of notes.*
Answers without justification will be regarded skeptically.

1. (25 points) The following statements (a)-(e) are false. For each statement, give a concrete counterexample to demonstrate its falsehood.

(a) If A is a 2×2 matrix where A^2 is the zero matrix, then A is the zero matrix.

(b) If A is not invertible, then A is not diagonalizable.

(c) If a linear transformation $T : V \to W$ maps V onto W, then T is one-to-one.

(d) If A is a matrix with $\dim \operatorname{Nul} A = 0$, then $\dim\, (\operatorname{Col} A)^{\perp} = 0$ as well.

(e) If $\vec{f_1}(t), \vec{f_2}(t)$ are continuous vector-valued functions, and if t_0 is a real number where $\vec{f_1}(t_0), \vec{f_2}(t_0)$ are linearly dependent, then the two functions themselves are linearly dependent.

2. (20 points)

(a) Compute the first column of the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 8 \end{pmatrix}$.

(b) Compute the determinant of $\begin{pmatrix} 2 & 0 & 4 \\ -2 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}$.

(c) Compute the rank of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 7 \end{pmatrix}$. Give the dimension of the nullspace as well.

(d) Compute the eigenvalues and eigenvectors of $\begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}$. Give the answer as a diagonalization.

3. (20 points) The vector space $\mathbb{P}_2$ is the set of polynomials whose degree is at most two. We define $\langle p(x), q(x)\rangle = \int_0^1 xp(x)q(x)\,dx$, which you may assume is an inner product on $\mathbb{P}_2$. (Make sure to notice the extra x in the integral, and remember $\int_0^1 x^n\,dx = \frac{1}{n+1}$.)

(a) Calculate an orthogonal basis for the subspace $\mathrm{Span}\{1, x\}$ by performing the Gram-Schmidt process.

(b) Write down the formula for the projection of x^2 onto the subspace $\mathrm{Span}\{1, x\}$. Compute the inner products which appear in it, but do not simplify the expression any further.

4. (20 points) Let U be an $n \times n$ orthogonal matrix which is also symmetric. Prove that if $U - I_n$ is an invertible matrix, then $U = -I_n$.

5. (15 points)

(a) Compute the general solution to $y''' + 2y'' + 2y' = 0$.

(b) Find all solutions $y(t)$ which do *not* converge to 0 as $t \to \infty$, and specify what each of these solutions converges to.

6. (20 points)

(a) Compute the general solution to $\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ -12 & 1 \end{pmatrix} \vec{x}(t)$.

(b) (No justification required.) Is there a homogeneous solution $\vec{x}(t)$ where, when $t \to \infty$,

(i) $\|\vec{x}(t)\| \to \infty$? YES / NO (ii) $\|\vec{x}(t)\| \to 0$? YES / NO *(circle your responses)*

(c) Compute a particular solution to $\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ -12 & 1 \end{pmatrix} \vec{x}(t) + \begin{pmatrix} 2 \\ 12 \end{pmatrix} e^t$ using any method.

7. (20 points) Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) What is the dimension of the solution set to the homogeneous system $\vec{x}'(t) = A\vec{x}(t)$?

(b) Compute the general solution to the homogenous system $\vec{x}'(t) = A\vec{x}(t)$.