Discussion – June 24

Transformations

1. Find the matrices for the following transformations $\mathbb{R}^2 \to \mathbb{R}^2$.
   (a) Rotation by $90°$ CW (b) Rotation by $180°$
   (c) Reflection about x-axis (d) Reflection about y-axis
   (e) Reflection about x-axis followed by reflection about y-axis.

2. Illustrate the action on $\mathbb{R}^2$ of $T(\vec{x}) = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} \vec{x}$.
   (Draw how it transforms a simple picture.)

3. If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(\vec{x}) = \vec{x}$, what is $[T]$?

Independence

1. Either find four vectors in $\mathbb{R}^3$ which are independent, or explain why it cannot be done.
2. Do the same for three vectors in $\mathbb{R}^3$.
3. Do the same for four vectors in $\mathbb{R}^5$.
4. $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b} \in \mathbb{R}^n$ are linearly independent. Is the system $[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3 \mid \vec{b}]$ consistent?

Transformations II

1. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, and $T(\vec{x}) = A\vec{x}$ and $S(\vec{x}) = B\vec{x}$, Compute $[T \circ S]$. Here, $\circ$ means function composition: $(T \circ S)(\vec{x}) = T(S(\vec{x}))$.
   (The idea is to derive matrix multiplication from the standard matrix of the composition of transformations. That is, $[T \circ S] = [T][S] = AB$.)

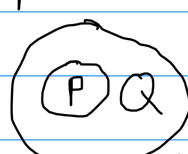
the beginnings of a list... (A is $m \times n$)

- existence (≥ 1 solution to $A\vec{x} = \vec{b}$ for all $\vec{b}$)
- A has pivot in every row (so $m \leq n$)
- The columns of A span $\mathbb{R}^m$
- $T(\vec{x}) = A\vec{x}$ is onto (surjective)
- Every $\vec{b} \in \mathbb{R}^m$ is a linear combination of $\vec{a}_1, \ldots, \vec{a}_n$
- If $\vec{b} \in \mathbb{R}^m$, $\vec{b} \in \operatorname{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

- uniqueness (≤ 1 solution to $A\vec{x} = \vec{b}$ for all $\vec{b}$)
- $A\vec{x} = \vec{0}$ has only the trivial solution
- A has a pivot in every column (so $n \leq m$)
- The columns of A are linearly independent
- $T(\vec{x}) = A\vec{x}$ is one-to-one (injective)
- Whenever $x_1, \ldots, x_n \in \mathbb{R}$ make $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = 0$, actually $x_1 = \cdots = x_n = 0$.

<u>Logic</u> P and Q are statements which are either true or false.

- if P then Q $\equiv$ Q if P $\equiv$ P only if Q $\equiv$ $\underbrace{\text{if not Q then not P}}_{\text{"contrapositive"}}$

  Venn diagram: 

  ($\equiv$ Q or not P)

  "When in the land of P being true, also in the land of Q being true."

- P if and only if Q $\equiv$ (P if Q) and (P only if Q) "P and Q logically equivalent"

  Venn diagram:  (same circle)