MATH 54 QUIZ II, KYLE MILLER
APRIL 12, 2016, 32 MINUTES

1. (5 points) Let $A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$. Find the least-squares solution(s) to $A\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

The least squares solutions are the solutions to $A^TA\hat{x} = A^T\vec{b}$. We compute A^TA and $A^T\vec{b}$:

$$A^TA = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 10 \end{pmatrix}$$

$$A^T\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}.$$

Thus, $\hat{x} = \begin{pmatrix} 5/9 \\ 1/9 \\ 1/5 \end{pmatrix}$ is the least-squares solution.

Or, noticing that the columns of A are orthogonal, and recalling that the least-squares solutions correspond to projections onto $\operatorname{Col} A$, we compute the coordinate of $\vec{b}$ relative to A. Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be the columns of A.

$$c_1 = \frac{\vec{v}_1 \cdot \vec{b}}{\vec{v}_1 \cdot \vec{v}_1} = \frac{5}{9}$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{b}}{\vec{v}_2 \cdot \vec{v}_2} = \frac{1}{9}$$

$$c_3 = \frac{\vec{v}_3 \cdot \vec{b}}{\vec{v}_3 \cdot \vec{v}_3} = \frac{2}{10}$$

These coefficients are such that $A\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \operatorname{proj}_{\operatorname{Col} A}\vec{b}$, so $\hat{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$.

2. (5 points) Consider the differential equation $y'' + 2y' + 2y = \cos(t)$.

a. Find the general solution to this differential equation.

First, we compute the homogeneous solution. The characteristic polynomial is $r^2 + 2r + 2 = 0$, which has roots $r = \frac{-2\pm\sqrt{4-4\cdot 2}}{2} = -1 \pm i$. Thus, homogenous solution is $y_h = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$.

Second, we compute a particular solution. The inhomogeneity $\cos t$ corresponds to the roots $r = \pm i$, which are not $-1 \pm i$, thus $y_p = A\cos t + B\sin t$ is a reasonable guess for the method of undetermined coefficients. We must calculate A and B by enforcing the equality $y_p'' + 2y_p' + 2y_p = \cos t$. For this, we find y_p' and y_p'':

$$\begin{aligned} y_p &= A\cos t + B\sin t \\ y_p' &= -A\sin t + B\cos t \\ y_p'' &= -A\cos t - B\sin t \end{aligned}$$

We have $y_p'' + 2y_p' + 2y_p = (-A+2B+2A)\cos t + (-B-2A+2B)\sin t = (A+2B)\cos t + (-2A+B)\sin t$, so we obtain the system of equations $A + 2B = 1$ and $-2A + B = 0$ for this to equal $\cos t$. Solving, $A = \frac{1}{5}$ and $B = \frac{2}{5}$. Thus, the general solution to the differential equation is

$$y = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t.$$

b. Solve the initial value problem $y(0) = 0$ and $y'(0) = 0$.

We need to find C_1 and C_2 so that the initial conditions are satisfied. First we compute y':

$$y' = -C_1 e^{-t}\cos t - C_1 e^{-t}\sin t - C_2 e^{-t}\sin t + C_2 e^{-t}\cos t - \frac{1}{5}\sin t + \frac{2}{5}\cos t.$$

Then, checking the initial conditions:

$$\begin{aligned} 0 &= y(0) = C_1 + 0 + \frac{1}{5} + 0 \\ 0 &= y'(0) = -C_1 - 0 - 0 + C_2 - 0 + \frac{2}{5} \end{aligned}$$

Thus, $C_1 = -\frac{1}{5}$ and $C_2 = C_1 - \frac{2}{5} = -\frac{3}{5}$. This gives the following solution to the initial value problem:

$$y = -\frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + \frac{1}{5}\cos t + \frac{2}{5}\sin t.$$

3. (5 points) The 2×2 matrix A is symmetric with two eigenvalues 2 and 1. One eigenvector of A is $\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$ with the corresponding eigenvalue 2. Compute A. (Hint: consider using spectral decomposition.)

Since A is a symmetric matrix, the spectral theorem applies: eigenspaces are mutually orthogonal, so an eigenvector with eigenvalue 2 and an eigenvector with eigenvalue 1 are orthogonal. The vector $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ is orthogononal to $\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$, so, since A has 1 as an eigenvalue, this vector must be an eigenvector with eigenvalue 1.

By spectral decomposition, $A = 2u_1u_1^T + u_2u_2^T$ where u_1 and u_2 are unit-length eigenvectors of eigenvalue 2 and 1, respectively. The lengths of the two aforementioned vectors are both $\sqrt{5}$, so we have

$$\begin{aligned} A &= 2\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \end{pmatrix}\right) + \left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & 2 \end{pmatrix}\right) \\ &= \frac{2}{5}\begin{pmatrix} 2 \\ -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} \\ &= \frac{2}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 9/5 & -2/5 \\ -2/5 & 6/5 \end{pmatrix}. \end{aligned}$$

(Then to check our work, verify that $1, 2$ are eigenvalues, with $\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$ an eigenvector with eigenvalue 2.)

Alternatively, we may use orthogonal diagonalization of A by constructing an orthogonal matrix P of eigenvectors. If we make use of $P^{-1} = P^T$, we must make sure the columns are orthonormal, and not just orthogonal! Thus, $A = PDP^T$ with

$$\begin{aligned} D &= \begin{pmatrix} 2 & \\ & 1 \end{pmatrix} \\ P &= \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}. \end{aligned}$$

We can avoid the $\sqrt{5}$ by doing plain old diagonalization $A = PDP^{-1}$ without worrying about orthogonality of P (note the $^{-1}$) with

$$P = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix},$$

whose inverse is

$$P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

If you do not like multiplying matrices, you can also construct the following system:

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$
$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

and solve for a, b, c.

If you did not realize the eigenvectors must be orthogonal, this becomes a rather difficult problem.

For fun. (0 points) (Inessential motivation: a falling particle is subject to wind resistance proportional to its velocity. What is its terminal velocity?) Consider $y'' + cy' = g$, where c and g are positive constants.

(a) Compute the general solution to this differential equation.

The homogeneous equation's characteristic polynomial is $r^2 + cr = 0$, which has roots $r = 0, -c$. Thus, the homogeneous solution is $y_h = C_1 e^{0t} + C_2 e^{-ct} = C_1 + C_2 e^{-ct}$.

Since $g = ge^{0t}$, it corresponds to the root 0, so it is virtually a double root. A reasonable guess for the particular solution is then $y_p = Ate^{0t} = At$. We have $y_p' = A$ and $y_p'' = 0$, so $y_p'' + cy_p' = g$ gives $cA = g$, and so $A = \frac{g}{c}$.

Therefore, the general solution is $y = C_1 + C_2 e^{-ct} + \frac{g}{c} t$.

(b) Compute $\lim_{t\to\infty} y'(t)$. Does this depend on initial conditions?

The derivative to the general solution is $y' = -cC_2 e^{-ct} + \frac{g}{c}$. Thus, since $\lim_{t\to\infty} e^{-ct} = 0$, we have $\lim_{t\to\infty} y' = \frac{g}{c}$. This does not depend on initial conditions!

(To check your answer: how would a physicist solve this problem with only a free body diagram?)

The particle has two forces acting on it: a downward force g and an upward force cy' (i.e., a force proportional to velocity). The equilibrium (which is zero net force) occurs when $y' = \frac{g}{c}$. When this happens, the particle has constant velocity, and we can check stability by noting that if the velocity were slightly faster, the cy' force would become greater, slowing the particle down, and if the velocity were slightly slower, the cy' force would lessen, speeding the particle up. Thus, this is a stable equilibrium. (Also notice that $y'' = g - cy' = c(\frac{g}{c} - y')$, which is positive for velocities less than $\frac{g}{c}$ and negative for velocities greater than $\frac{g}{c}$. This is negative feedback.)