MATH 54 QUIZ I, KYLE MILLER
MARCH 1, 2016, 40 MINUTES
(5 PAGES)

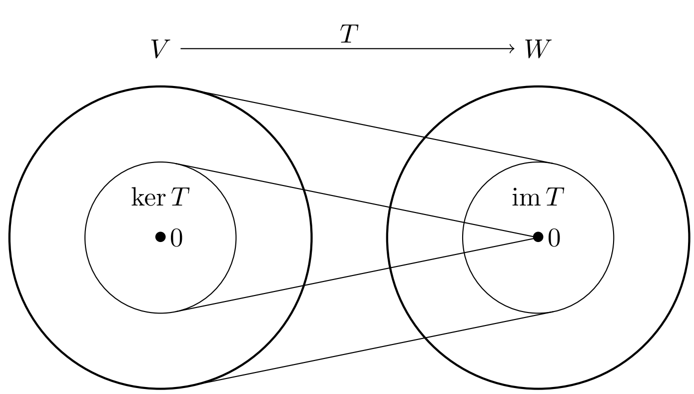
Problem Number	1	2	3	4	Total
Score					

YOUR NAME: SOLUTIONS

No calculators, no references, no cheat sheets.
Answers without justification will receive no credit.

Glossary

$\ker T$**:** the *kernel* of a linear transformation T.
$\operatorname{im} T$**:** the *image* or *range* of a linear transformation T.
onto**:** for $T : V \to W$, $\operatorname{im} T = W$.
one-to-one**:** for $T : V \to W$, $\ker T = \{0\}$.
basis**:** a linearly independent spanning set.
dimension**:** the number of vectors in a basis for a vector space.



1. (6 points) For each of the following, find all values of $a \in \mathbb{R}$ (if any) so that the given set of vectors spans $\mathbb{R}^3$.

(a) (2 points)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} \right\}$$

This is a set of two vectors, but fewer than three vectors never spans $\mathbb{R}^3$. Hence, there are no a.

In other words, to span $\mathbb{R}^3$, the equation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & a \end{pmatrix} \vec{x} = \vec{b}$$

must be solvable for all $\vec{b} \in \mathbb{R}^3$. However, there are only two pivots, not three.

(b) (2 points)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 0 \\ 1 \end{pmatrix} \right\}$$

Using the matrix idea from before, there are always three pivots no matter the choice fo a. Thus, the three vectors always span $\mathbb{R}^3$, for all $a \in \mathbb{R}$.

(b) (2 points)

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ a \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix} \right\}$$

We row-reduce the matrix of vectors:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 3 & 1 & a & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 4 & a-3 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & a-11 & 0 \end{pmatrix}$$

So, if $a = 11$, there are only two pivots (in which case the vectors do not span $\mathbb{R}^3$), but if $a \neq 11$, there are three pivots, and the vectors span $\mathbb{R}^3$.

2. (5 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\vec{x}) = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \vec{x}.$$

(a) (2 points) Find a basis for $\operatorname{im} T$.

For transformations between Euclidean vector spaces, such as T, the image is the column space of its standard matrix, and to find a basis for $\operatorname{Col}[T]$, we row-reduce the matrix and take columns from $[T]$ corresponding to the pivot columns. Row-reducing:

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, a basis for $\operatorname{im} T$ is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) (2 points) Find a basis for $\ker T$.

Since T is a transformation between Euclidean vector spaces, we can just find a basis for $\operatorname{Nul}[T]$. We already have the reduced row-echelon form of $[T]$, so we solve $[T]\vec{x} = \vec{0}$ by noting the third column is free, so all solutions are of the form $x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ with $x_3 \in \mathbb{R}$. Therefore, a basis is

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}.$$

(c) (1 point) Find a linear transformation $F : V \to \mathbb{R}^3$ whose image is $\ker T$, and where F is one-to-one. You get to choose the vector space V.

A couple options:

(1) Let $V = \ker T$ and $F : \ker T \to \mathbb{R}^3$ be defined by $F(\vec{x}) = \vec{x}$. This F is a linear transformation (since $F(\vec{x} + \vec{y}) = \vec{x} + \vec{y} = F(\vec{x}) + F(\vec{y})$ and $F(c\vec{x}) = c\vec{x} = cF(\vec{x})$). It is one-to-one since $\ker F = \{\vec{x} \in \ker T : F(\vec{x}) = \vec{0}\} = \{\vec{x} \in \ker T : \vec{x} = \vec{0}\} = \{\vec{0}\}$, and $\operatorname{im} F = \ker T$ since $\operatorname{im} F = \{F(\vec{x}) : \vec{x} \in \ker T\} = \{\vec{x} : \vec{x} \in \ker T\} = \ker T$.

(Whenever W is a subspace of V, then the map $\iota : W \to V$ defined by $\iota(x) = x$ is called an "inclusion map" since the vectors of W are included into V. This makes sense since $W \subset V$.)

(2) Let $V = \mathbb{R}$ and $F : \mathbb{R}^1 \to \mathbb{R}^3$ defined by $F(x) = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} x$, which is a linear transformation because F is defined by a matrix. Then, $\operatorname{im} F = \ker T$ since $\operatorname{im} F$ is the span of the columns of the matrix, and the columns span $\ker T$. And F is one-to-one since vectors in the columns of the matrix are independent (since there is only one vector, and it is nonzero).

3. (6 points) $\mathbb{P}_2$ is the vector space of polynomials of degree at most two, with real coefficients.

(a) (3 points) Let S be the set of all polynomials from $\mathbb{P}_2$ whose derivative at 0 is 0 (that is, $p'(0) = 0$). Show that S is a vector subspace of $\mathbb{P}_2$.

A few ways to do this:

(1) We check the three properties for a subspace. First, it has the zero polynomial $p(x) = 0$ since the derivative of this polynomial at 0 is 0. Second, when p, q are two polynomials in $\mathbb{P}_2$ whose derivatives at 0 are 0, then $(p+q)'(0) = (p'+q')(0) = p'(0)+q'(0) = 0+0 = 0$, so $p+q$ is also a polynomial whose derivative at 0 is 0. Third, when p is a polynomial in $\mathbb{P}_2$ whose derivative at 0 is 0 and $c \in \mathbb{R}$, then $(cp)'(0) = cp'(0) = c \cdot 0 = 0$, so cp is also a polynomial whose derivative at 0 is 0. (By the way, we know $p+q$ and cp are in $\mathbb{P}_2$ because $\mathbb{P}_2$ is a vector space.)

(2) Define $T : \mathbb{P}_2 \to \mathbb{R}$ by $T(p) = p'(0)$. Then, $S = \ker T$, and kernels are always subspaces.

(3) Consider an arbitrary polynomial $p(x) = ax^2 + bx + c$ from $\mathbb{P}_2$. Then $p'(x) = 2ax + b$, and the requirement $p'(0) = 0$ amounts to saying $2a \cdot 0 + b = 0$, so $b = 0$. Thus, $S = \operatorname{Span}\{1, x^2\}$, and spans are always subspaces.

(b) (1 point) What is the dimension of S?

By doing the calculation in option 3 from the first part, we see $S = \operatorname{Span}\{1, x^2\}$, and since these two polynomials are linearly independent (they are elements of the "standard basis" for $\mathbb{P}$ after all), we see S has a basis of two polynomials. Hence, $\dim S = 2$.

Or, using option 2, notice the image of T is all of $\mathbb{R}$ (for instance, $T(cx) = c$ for all c), so $\dim \operatorname{im} T = 1$. Since $\dim \mathbb{P}_2 = 3$, and since $\dim \operatorname{im} T + \dim \ker T = \dim \mathbb{P}_2$, we have $\dim S = \dim \ker T = 3 - 1 = 2$.

(c) (2 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be defined by $T(p) = p(x-1) - p(x)$. (For instance, $T(x^2+1) = ((x-1)^2 + 1) - (x^2+1)$.) What are $\ker T$ and $\operatorname{im} T$? Describe them by finding a basis for each.

To see what is going on, we calculate

$$\begin{aligned} T(ax^2 + bx + c) &= (a(x-1)^2 + b(x-1) + c) - (ax^2 + bx + c) \\ &= ax^2 - 2ax + a + bx - b + c - ax^2 - bx - c \\ &= -2ax + a - b. \end{aligned}$$

For $\ker T$, we are solving $T(ax^2 + bx + c) = 0$, so solving $-2ax + a - b = 0$. Then, $a = 0$ and $b = 0$, which leaves c free, so $\ker T = \{c \in \mathbb{R}\} = \operatorname{Span}\{1\}$.

For $\operatorname{im} T$, we are finding all possible values $T(ax^2+bx+c)$, so finding all possible values $-2ax+a-b = a(-2x+1) - b$. This is $\operatorname{Span}\{-2x+1, 1\}$, which is the same as $\operatorname{Span}\{x, 1\}$ after replacement and scaling (either of these two is a fine basis).

4. (5 points) Let A be an $n \times m$ matrix and B be an $m \times n$ matrix such that $BA = I_m$.

(a) (2 points) What is the dimension of $\operatorname{Col} B$?

Intuition: $\operatorname{Col} B$ has to be the same as the column space of I_m. Basically, where would the m pivots come from?

Beware: $BA = I_m$ does not mean A or B are invertible. For instance,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \end{pmatrix}.$$

A way to solve this is to first figure out $\operatorname{Col} B$. Remember $\operatorname{Col} B$ is all $B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Let $\vec{y} \in \mathbb{R}^m$. Then, $BA\vec{y} = I_m\vec{y} = \vec{y}$. So, for every $\vec{y} \in \mathbb{R}^m$, then $B(A\vec{y}) = \vec{y}$ (and $A\vec{y} \in \mathbb{R}^n$ is one of those $\vec{x}$ mentioned above). This means $\operatorname{Col} B = \mathbb{R}^m$. Hence, $\dim \operatorname{Col} B = m$.

Or, in other words, for a vector $\vec{y} \in \mathbb{R}^m$, we have $BA\vec{y} = I_m\vec{y}$. This gives us $B(A\vec{y}) = \vec{y}$, so whenever we want to solve $B\vec{x} = \vec{y}$, we may as well let $\vec{x} = A\vec{y}$. This implies the columns of B span $\mathbb{R}^m$, so $\operatorname{Col} B = \mathbb{R}^m$.

(b) (2 points) What is the dimension of $\operatorname{Nul} A$?

Let us figure out $\operatorname{Nul} A$. Let $\vec{y} \in \mathbb{R}^m$ be a vector where $A\vec{y} = \vec{0}$. Then $BA\vec{y} = B\vec{0}$, which simplifies to $I_m\vec{y} = \vec{0}$, and so $\vec{y} = 0$. This means the only vector in $\operatorname{Nul} A$ is the zero vector. Thus, $\dim \operatorname{Nul} A = 0$.

(c) (1 point) Which of the following cannot happen? $n > m$ or $m > n$? Explain why not.

What cannot happen is $m > n$. Two reasons, either is sufficient by itself:

(1) If $m > n$, then B would have more rows than columns, which by a theorem from the book means the columns of B do not span $\mathbb{R}^m$, which means $\operatorname{Col} B \neq \mathbb{R}^m$, contradicting part (a).
(2) If $m > n$, then A would have more columns than rows, forcing A to have free columns, which would mean $\operatorname{Nul} A$ is nontrivial, contradicting part (b).

For fun. (0 points) Let A be an $n \times n$ matrix such that $A^2 = A$. Which vectors are in both $\operatorname{Col} A$ and $\operatorname{Nul} A$?

We will show that the only vector in both $\operatorname{Col} A$ and $\operatorname{Nul} A$ is the zero vector. Let v be a vector which is in both. Since $v \in \operatorname{Col} A$, v is a linear combination of the columns of A, so there is some $x \in \mathbb{R}^n$ such that $v = Ax$. Since $v \in \operatorname{Nul} A$, we have $Av = 0$. Now, apply A to both sides of $v = Ax$ to get $Av = A^2x$. Since $Av = 0$, this becomes $0 = A^2x$, and since $A^2 = A$, this becomes $0 = Ax$ (implying $x \in \operatorname{Nul} A$, too). Since $v = Ax$, then $v = 0$. Therefore, the only vector in both is the zero vector.