Another solution for midterm 2's problem 4

Kyle Miller

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Probably because of Spring break, I completely forgot to show you a much better solution to the fourth problem on the midterm today. This is definitely what Prof. Yuan had in mind when he created the exam, since Gram-Schmidt takes too much work.

To remind you, this is problem 4 in full:

> 4. (5 points) Let W be the subspace of $\mathbb{R}^4$ given by $x_1 + x_2 + x_3 + x_4 = 0$. Compute the orthogonal projection $\operatorname{proj}_W(\vec{v})$ of the vector $\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ to the space W.

First observation: $\operatorname{proj}_W \vec{v} + \operatorname{proj}_{W^\perp} \vec{v} = \vec{v}$. If we can calculate $\operatorname{proj}_{W^\perp} \vec{v}$ easily, then it is a quick calculation to get $\operatorname{proj}_W \vec{v}$. It should be easy: $\dim W^\perp = 1$ since $\dim W = 3$ (which is because W is the nullspace of $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$), so $W^\perp$ is the span of a single vector.

Now,

$$W^\perp = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Here are a couple of justifications:

- Recall that $(\operatorname{Nul} A)^\perp = \operatorname{Row} A$. $W^\perp = \left(\operatorname{Nul} \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}\right)^\perp = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

- Saying that $x_1 + x_2 + x_3 + x_4 = 0$ is the same as saying $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \vec{x} = 0$, so W is already $\left(\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}\right)^\perp$. Then $W^\perp = \left(\left(\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}\right)^\perp\right)^\perp = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Let $\vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$. Then $\text{proj}_{W^\perp} \vec{v} = \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{10}{4}\vec{u} = \frac{5}{2}\vec{u}$.

Thus, $\text{proj}_W \vec{v} = \vec{v} - \text{proj}_{W^\perp} \vec{v} = \vec{v} - \frac{5}{2}\vec{u} = \begin{pmatrix} -3/2 \\ -1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$.

1 Another solution to problem 6

For the 3:30–5:00pm section, I gave a solution to problem 6 involving Gauss-Jordan elimination. This was not a good solution, and what I said then should be disregarded. The G-J elimination solution works much better like this:

- Comparing $\text{rank}(AB)$ and $\text{rank}(A)$ is the same as comparing $\text{rank}(B^T A^T)$ and $\text{rank}(A^T)$, since transposition does not change rank.
- Since B is invertible, so is B^T, so the reduced row-echelon form of B^T is I_n.
- The sequence of row operations correspond to elementary matrices, hence $B^T = E_1 \cdots E_n$, a product of elementary matrices.
- Then $B^T A^T = E_1 \cdots E_n A^T$.
- But this is row-equivalent to A^T since the difference is a bunch of elementary row operations (represented by $E_1, \ldots, E_n$).
- That is, $B^T A^T \sim A^T$.
- Thus $\text{rank}(B^T A^T) = \text{rank}(A^T)$.
- Therefore $\text{rank}(AB) = \text{rank}(A)$.