Discussion notes

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Now that we have familiarity with $\mathbb{R}^n$, we can try to think about exactly which properties about $\mathbb{R}^n$ are the ones we most like. Mathematicians have decided that addition and scalar multiplication, along with some coherence properties, are the essential properties of a vector space:

Definition 1. *A (real)*[1] vector space *is the following ingredients:*

- *a nonempty set V of* vectors*;*[2]
- *an addition operation $+ : V \times V \to V$;*[3]
- *a scalar multiplication operation $\mathbb{R} \times V \to V$;*
- *the following properties: (assume $u, v, w \in V$ and $c, d \in \mathbb{R}$)*
  1. *There is an additive identity $0 \in V$.*[4] *(That is, $0 + v = v + 0 = v$.)*
  2. *Addition is commutative. (That is, $v + w = w + v$.)*
  3. *Addition is associative. (That is, $u + (v + w) = (u + v) + w$. This amounts to saying that you don't need parentheses for addition. Compare to subtraction.)*
  4. *There are additive inverses. (That is, there is a vector $-v$ for each v such that $v + (-v) = (-v) + v = 0$. When we write $v - w$, we actually mean $v + (-w)$.)*
  5. *Addition and scalar multiplication distribute. (That is, $c(v + w) = cv + cw$.)*
  6. *Addition of scalars and scalar multiplication distribute. (That is, $(c + d)v = cv + dv$.)*
  7. *Scalar multiplication is an action with respect to multiplication of scalars. (That is, $c(dv) = (cd)v$.)*
  8. *$1 \in \mathbb{R}$ is the identity action. (That is, $1v = v$).*

[1] vs. complex; that is, $\mathbb{R}$ vs $\mathbb{C}$ for the set of scalars.

[2] Thus a "vector" is determined by which vector space you're talking about.

[3] This is read as "plus is an operation taking a pair of vectors in V to a vector in V." The book talks about "closure of addition," but the definition here amounts to the same thing.

[4] We use 0 to denote the additive identity no matter the vector space.

It is mathematical convention to use synecdoche and metonymy whenever it makes saying things simpler, which is a practice neophytes can find somewhat confusing. So, while a vector space is all of these things together (a set of vectors, and addition operation, a scalar multiplication, and some properties these three must together have), we tend to call the set of vectors the vector space, assuming the reader or listener is able to understand from context the addition and scalar multiplication. For instance, we say $\mathbb{R}^n$ is a vector space, assuming the standard addition and scalar multiplication. If it's unclear which vector space the set of vectors might be standing in for, you can always ask something like "what's the addition operation?"

It is a good exercise to prove basic things about general vector spaces that are true about all vector spaces. Here are some examples.[5]

1. *For every vector $u \in V$, $0u = 0$. (The first 0 is $0 \in \mathbb{R}$, and the second 0 is $0 \in V$. I'm not using vector hats so that you get used to it.)*

   $0 = 0u + (-(0u))$ by property 4. Using the fact that $0 + 0 = 0$ in $\mathbb{R}$, we can see $0u = (0+0)u$, so substituting we have $0 = (0+0)u + (-(0u))$, and by property 6, this is $0 = (0u + 0u) + (-(0u))$. By associativity, this is $0 = 0u + (0u + (-(0u))$, and by property 4, this is $0 = 0u + 0$, which is $0 = 0u$ by property 1.

2. *For every scalar $c \in \mathbb{R}$, $c0 = 0$.*

   You can figure out which properties we use in this: $0 = c0 + (-(c0)) = c(0+0) + (-(c0)) = c0 + c0 + (-(c0)) = c0 + 0 = c0$. Hence, $0 = c0$.

3. *For every $u \in V$, $-u = (-1)u$. (Note carefully, we are saying the additive inverse is equal to scaling by -1. Hence, from now on, we can pretend these two different operations are actually the same operation.)*

   Using $0 = 0u$, we have $-u = -u + 0 = -u + (1-1)u = -u + 1u + (-1)u = -u + u + (-1)u = 0 + (-1)u = (-1)u$.

Other things you can prove: that there is only one additive identity (so if 0 and $0'$ are both additive identities, prove $0 = 0'$), that there is only one additive inverse (so if u and u' are both additive inverses of v, then $u = u'$), and if $cv = v$ then either $c = 1$ or $v = 0$.

The following are examples of vector spaces. For each of them, decide what addition and scalar multiplication are and check to the best of your ability that each of the required eight properties hold, and, for instance, what is the zero vector:

- $\mathbb{R}$ with its normal addition and multiplication
- (non-example) $\mathbb{R}$ with vector addition being multiplication
- $\mathbb{R}^n$ as column vectors

[5] The way I do them might seem a little awkward; that is because I don't want to have to reason about doing things to both sides being reversible. Try coming up with your own arguments from the properties in a way you find pleasing.

- Arrows in 3D space, modulo translation (this means that two vectors are equivalent if they are equal after a suitable translation. Modulo is a good word to know: for instance, two numbers are the same modulo n if they are different by a multiple of n.)

- The set of convergent sequences. Be sure to check that addition of convergent sequences is convergent, and that scalar multiplication of a convergent sequence is convergent.

- The set of polynomials (denoted $\mathbb{P}$).

- The set of polynomials of degree at most n (denoted $\mathbb{P}$). The degree of a polynomial is the largest n in any cx^n present in the polynomial, where $c \neq 0$. We presume the zero polynomial always has degree at most n.

- The set of real-valued continuous function $\mathcal{C}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is continuous}\}$.

- The set of real-valued infinitely differentiable (i.e., smooth) functions $\mathcal{C}^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} | \frac{d^n}{dx^n} f(x) \text{ continuous for all } n\}$.

- The set of all power series $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence nonzero.

These are all legitimate examples of vector spaces that are actually used. Not one is pathological!

In algebra, the way things always work is you define algebraic objects (in this case vector spaces), maps between them (linear transformations; though we haven't defined them for vector spaces, they are essentially the same as before), and some sort of *sub-* object, which is in our case a *vector subspace.*

Definition 2. *A* vector subspace *(or* subspace *for short) is a subset W of a vector space V which is a vector space which inherits addition and scalar multiplication from V. This means for $u, v \in W$, $u + v$ (with addition in W) equals $u + v$ (with addition in V), and similarly for scalar multiplication.*

A useful set notation is $W \subset V$ to mean W is a subset of V. When I use the symbol, this means W might equal V. When I want a *proper* subset, which is a subset that doesn't equal V, I write $W \subsetneq V$. Sometimes, others write $W \subset V$ for proper subset, and $W \subseteq V$ for when W might equal V. Confusing.

There is a convenient way to check whether something is a vector subspace. In fact, the book just jumps to these conditions being the definition, though I prefer thinking of the following as a theorem:

Theorem 1. *For a subset W of a vector space V, the following conditions are equivalent to W being a subspace of V:*

- $0 \in W$.

- *W has closure under addition (that is, for $u, v \in W$, then $u + v \in W$).*

- *W has closure under scalar multiplication (that is, for $c \in \mathbb{R}$ and $u \in W$, then $cu \in W$).*

Just to get the idea down, remember the metonymy: we are saying that if these three conditions hold, then there is a vector space structure making W a vector space, with all of the right properties and addition and scalar multiplication (inherited in a straightforward way from V).

In practice, these three conditions are much easier to check than showing W is a subspace straight from the definition.

Here are some examples. For each of these, check each of the three conditions:

- For V a vector space, V is a vector subspace of V.
- For V a vector space, $\{0\} \subset V$ is a vector subspace of V (called the *zero subspace*).
- $\{(x_1, x_2, \ldots, x_{n-1}, 0) | x_1, \ldots, x_n \in \mathbb{R}\} \subset \mathbb{R}^n$ (the set of all vectors in $\mathbb{R}^n$ whose last component is 0).
- We have a chain of subspaces: $\mathbb{P}^n \subset \mathbb{P} \subset \mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{C}(\mathbb{R})$.
- Nonexample: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$. This is a category error. 2×1 matrices just aren't 3×1 matrices.
- Nonexample: $\{(x, y) | x + y = 1\} \subset \mathbb{R}^2$ (missing 0)
- Nonexample: $\{(x, 0) | x \in \mathbb{R}\} \cup \{(0, y) | y \in \mathbb{R}\} \subset \mathbb{R}^2$, the union of the x-axis and the y-axis (contains 0 but not closed under addition; consider e_1 and e_2).
- Nonexample: $\{(x, y) | x, y \in \mathbb{R}, x \geq 0\} \subset \mathbb{R}^2$ (contains 0 and is closed under addition, but not closed under scalar multiplication; consider $-e_1$).
- For U, W subspaces of V, consider the intersection $U \cap W$. This is a subspace because 1) $0 \in U$ and $0 \in W$, so $0 \in U \cap W$, 2) for $v_1, v_2 \in U \cap W$, then both vectors are in both U and W, so we can do addition in U and in W, where the result lies in U and W, so $v_1 + v_2 \in U \cap W$, 3) exercise.
- The span of $v_1, \ldots, v_n \in V$. This is an extremely important example; most of our effort will be going into describing subspaces as spans of some collection of vectors.