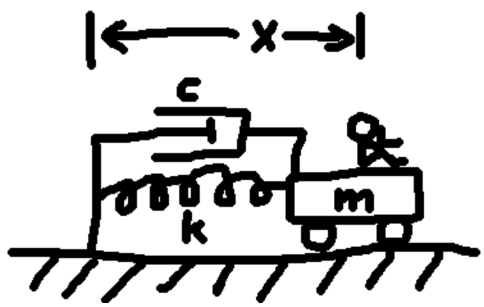
1 Phase diagrams

Systems of differential equations are used to model physical systems. In mechanics, the state of a system is a collection of all of the positions and momenta of the various objects under consideration. For instance, for a mass-spring system, the state is the position x and the momentum p of the mass. A phase diagram is a graph which represents how the derivatives of the state variables depend on the state. For a single object, horizontally lies the x axis, and vertically the p axis. At each state (x, p), we place the vector (x', p'). Sometimes we normalize the vectors if we do not care about the magnitudes (as we have done on the following page). Slope fields are a particular kind of phase diagram.



Consider a dampened mass-spring system. The Hooke's law spring force is $-kx$, and for our purposes damping will be proportional to velocity, so $-cv$. Then we have $p' = (mv)' = ma = -kx - cv$ and $x' = \frac{1}{m}p$, which gives the system

$$\begin{bmatrix} p' \\ x' \end{bmatrix} = \begin{bmatrix} -c & -k \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} p \\ x \end{bmatrix}$$

The characteristic equation of this matrix is $(-c - \lambda)(-\lambda) + \frac{k}{m} = \lambda^2 + c\lambda + \frac{k}{m}$.

Exercise 1. *When are the eigenvalues distinct, singular, or complex? Find the eigenvectors in each case. See the next page for what these correspond to.*

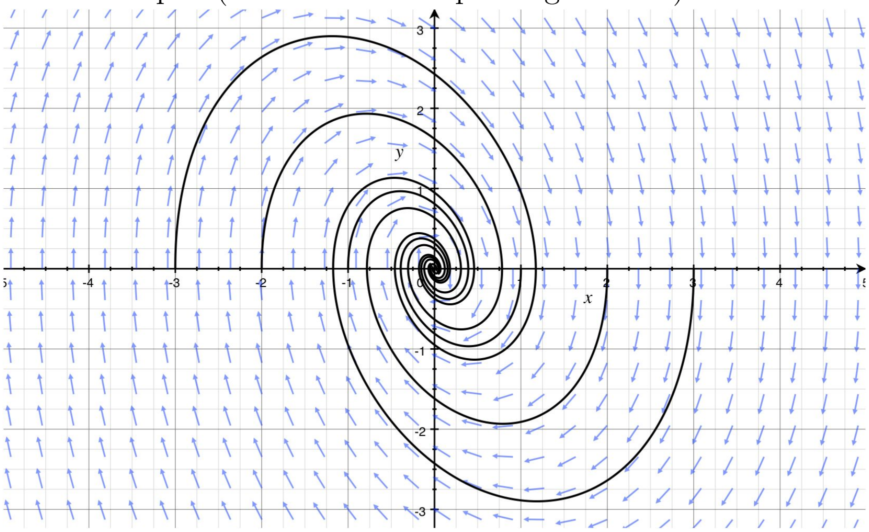
It is left as a further exercise to use these eigenvectors to solve the differential equation. For the following, we are going to plot phase diagrams of second-order linear differential equations using the correspondence $u = x$ and $v = x'$, plotting the vectors (u', v').

Exercise 2. *Plot phase diagrams for* $x'' - x' - 2x = 0$, $x'' + x' - 2x$, $x'' + x = 0$, $x'' - 4x' + 4x = 0$, $x'' - x' + x = 0$, *and* $x'' + x' + x = 0$. *How do the eigenvalues control the shapes of the diagrams?*

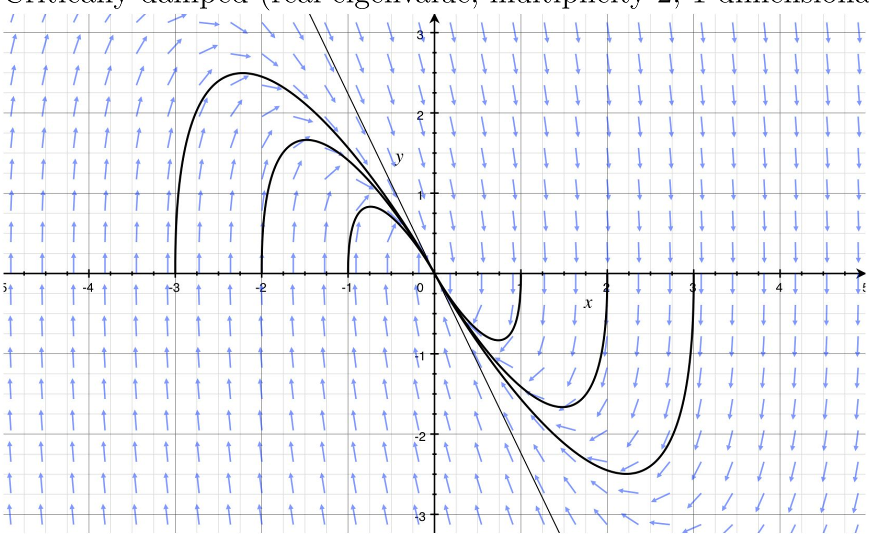
Connection with eigenvectors: Suppose $\vec{x}' = A\vec{x}$, and $A = PDP^{-1}$. Then $P^{-1}\vec{x}' = DP^{-1}\vec{x}$, and since $(B\vec{x}) = B\vec{x}'$ for any matrix B, then $(P^{-1}\vec{x})' = D(P^{-1}\vec{x})$. Since D is diagonal, this is very easy to solve for $\vec{u}' = D\vec{u}$, and then $\vec{x} = P\vec{u}$. When A is not diagonalizable, use Jordan Normal Form for A, with $A = PBP^{-1}$. Then $\vec{u}' = B\vec{u}$ is a system of differential equations which can be solved using integrating factors.

Though, the punchline is that you can just check whether $t^k e^{\lambda t}$ appears in the solution to $\vec{x}$ for $\vec{x}' = A\vec{x}$, where $0 \le k \le m_\lambda$, and where m_λ is the multiplicity of the eigenvalue λ.

Under damped (two distinct complex eigenvalues):



Critically damped (real eigenvalue, multiplicity 2, 1 dimensional eigenspace):



Overdamped (two real eigenvalues):