Quiz 9

1. (5 points) (a) Find a vector $\vec{v}_3 \in \mathbb{R}^3$ which is orthogonal to both $\vec{v}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.
(b) What are all the vectors orthogonal to $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$? (Hint: $\dim V^\perp$ for $V = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.)

(a) Finding a vector which is orthogonal to a collection of vectors is essentially solving a homogeneous linear system:

$$\begin{pmatrix} 2 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix} \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

(Compare with $(\text{Col}\, A)^\perp = \text{Nul}\, A^T$.) Row reduction of the coefficient matrix gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1/2 \end{pmatrix}$$

so a possible vector is $\vec{v}_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$.

(b) The hint was meaning for you to think about $\dim V + \dim V^\perp = 3$, so since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, $\dim V = 3$, hence $\dim V^\perp = 0$, and thus only $\vec{0}$ is orthogonal to all three vectors.
Alternatively, a vector which is orthogonal to all three vectors is in

$$\text{Nul} \begin{pmatrix} 2 & 2 & 2 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix},$$

whose matrix has three pivots, so the nullspace contains only $\vec{0}$.

2. (5 points) For $\vec{w} = \binom{3}{4}$ and $V = \text{Span}\{\binom{2}{1}\}$, find two vectors $\vec{w}^{\parallel} \in V$ and $\vec{w}^\perp \in V^\perp$ so that $\vec{w} = \vec{w}^{\parallel} + \vec{w}^\perp$. (In other words, decompose $\vec{w}$ into the projection/parallel component and the orthogonal component. The vector $\vec{w}^{\parallel}$ is $\text{proj}_V \vec{w}$.)

First we compute $\vec{w}^{\parallel} = \text{proj}_V \vec{w} = \frac{(3,4)\cdot(2,1)}{(2,1)\cdot(2,1)}(2,1) = \frac{10}{5}(2,1) = (4,2)$. Then, $\vec{w}^\perp = \vec{w} - \vec{w}^{\parallel} = (3,4) - (4,2) = (-1,2)$.
We check that $(-1,2)\cdot(2,1) = 0$, so $\vec{w}^\perp$ is indeed in the orthogonal complement. Also, $(3,4) = (4,2) + (-1,2)$.

(For fun) (a) Why is every square matrix a "root" of a polynomial? (For instance, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ satisfies $I + A + A^2 = 0$, so it is the "root" of $1 + x + x^2$.) (b) Let $p(x)$ be a polynomial where $p(A) = 0$. Why is every eigenvalue λ of A a root of p? (Hint: use an eigenvector $\vec{v}$ to show $p(A)\vec{v} = p(\lambda)\vec{v}$.) (c) If P is the matrix of proj_V, then $P^2 = P$ since projecting a projected vector does not change the vector. This means $P^2 - P = 0$. What are the possible eigenvalues of P?

(a) Note that $\mathbb{R}^{n\times n}$ is n^2-dimensional, so if we took $I, A, A^2, A^3, \ldots, A^{n^2}$, we have n^2+1 vectors, which is greater than n^2. Thus, there is some linear dependence $c_0I+c_1A+c_2A^2+\cdots+c_{n^2}A^{n^2} = 0$, and so A is a root of the polynomial $c_0 + c_1x + c_2x^2 + \cdots + c_{n^2}x^{n^2}$.
(b) Suppose A is a root of a polynomial $p(x)$, and suppose $\vec{v}$ is an eigenvector of A with eigenvalue λ. Then since $p(A) = 0$, we have

$$\begin{aligned} 0 &= p(A)\vec{v} \\ &= (c_0I + c_1A + \cdots + c_mA^m)\vec{v} \\ &= c_0\vec{v} + c_1A\vec{v} + \cdots + c_mA^m\vec{v} \\ &= c_0\vec{v} + c_1\lambda\vec{v} + \cdots + c_m\lambda^m\vec{v} \\ &= (c_0 + c_1\lambda + \cdots + c_m\lambda^m)\vec{v} \\ &= p(\lambda)\vec{v}, \end{aligned}$$

and since $\vec{v} \neq \vec{0}$, this implies $p(\lambda) = 0$, so λ is a root of $p(x)$.
(c) Since $P^2 - P = 0$, P is a root of $p(x) = x^2 - x = x(x-1)$, and p has roots $0, 1$. Thus, the only possible eigenvalues of P are 0 and 1. In fact, I is a matrix with only 1, the zero matrix is a matrix with only 0, and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is one with both.