Quiz 8

1. (5 points) Let $\vec{v}_1 = \left(\begin{smallmatrix}3\\1\end{smallmatrix}\right)$ and $\vec{v}_2 = \left(\begin{smallmatrix}5\\2\end{smallmatrix}\right)$. Find a 2×2 matrix A such that $A\vec{v}_1 = 2\vec{v}_1$ and $A\vec{v}_2 = -\vec{v}_2$.

The two facts about the matrix are that $\vec{v}_1$ is an eigenvector with eigenvalue 2 and $\vec{v}_2$ is an eigenvector with eigenvalue -1. This means that A is diagonalizable as $A = PDP^{-1}$ with

$$P = \begin{pmatrix}3 & 5\\ 1 & 2\end{pmatrix} \qquad D = \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}$$

Then, we compute

$$\begin{aligned} A = PDP^{-1} &= \begin{pmatrix}3 & 5\\ 1 & 2\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix} \\ &= \begin{pmatrix}6 & -5\\ 2 & -2\end{pmatrix}\begin{pmatrix}2 & -5\\ -1 & 3\end{pmatrix} \\ &= \begin{pmatrix}17 & -45\\ 6 & -16\end{pmatrix}. \end{aligned}$$

2. (5 points) For $A = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 2\\ 0 & 3 & 1\end{pmatrix}$, find the (a) eigenvalues, (b) eigenvectors, and (c) diagonalization.

(a) We calculate the characteristic polynomial:

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{pmatrix}-\lambda & 0 & 0\\ 0 & -\lambda & 2\\ 0 & 3 & 1-\lambda\end{pmatrix} \\ &= -\lambda(-\lambda(1-\lambda) - 6) \\ &= -\lambda(-\lambda + \lambda^2 - 6) \\ &= -\lambda(\lambda - 3)(\lambda + 2) \end{aligned}$$

so the eigenvalues are $0, 3, -2$. (Notice that there are three distinct eigenvalues, so the matrix is diagonalizable.)
(b) We compute bases for the eigenspaces. For $\lambda = 0$: (These are just vectors in the nullspace. I'll leave the row reductions to you.)

$$\text{Nul}(A - 0I) = \text{Nul}\,A = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$$

For $\lambda = 3$:

$$\text{Nul}(A - 3I) = \text{Nul}\begin{pmatrix}-3 & 0 & 0\\ 0 & -3 & 2\\ 0 & 3 & -2\end{pmatrix} = \text{Nul}\begin{pmatrix}1 & 0 & 0\\ 0 & -3 & 2\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}0\\2\\3\end{pmatrix}\right\}.$$

For $\lambda = -2$:

$$\text{Nul}(A+2I) = \text{Nul}\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 2\\ 0 & 3 & 3\end{pmatrix} = \text{Nul}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 0\end{pmatrix} = \text{Span}\left\{\begin{pmatrix}0\\ -1\\ 1\end{pmatrix}\right\}.$$

Thus, we have found three eigenvectors for the three eigenvalues. (It is wise to check that these are all actually eigenvectors.)
(c) $A = PDP^{-1}$ with

$$P = \begin{pmatrix}1 & 0 & 0\\ 0 & 2 & -1\\ 0 & 3 & 1\end{pmatrix} \qquad D = \begin{pmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -2\end{pmatrix}$$

(We can check this with $AP = PD$.)

(For fun) Why do A and A^T have the same eigenvalues?

Since $\det(A^T - \lambda I) = \det((A-\lambda I)^T) = \det(A - \lambda I)$, they have the same characteristic polynomials, so they have the same eigenvalues.