Quiz 7

1. (5 points) Suppose $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\}$ and $\mathcal{C} = \{\vec{c}_1, \vec{c}_2\}$ are two bases of some vector space V, with $\vec{b}_1 = 3\vec{c}_1 + 5\vec{c}_2$ and $\vec{b}_2 = \vec{c}_1 + 2\vec{c}_2$.

   (a) Find the change-of-coordinates matrix P from $\mathcal{B}$ to $\mathcal{C}$. (In other words, a P which satisfies $P[\vec{x}]_\mathcal{B} = [\vec{x}]_\mathcal{C}$ for all $\vec{x} \in V$.)

   (b) Find the change-of-coordinates matrix Q from $\mathcal{C}$ to $\mathcal{B}$.

   (c) What is QP equal to?

(a) The first column of P is $[\vec{b}_1]_\mathcal{C} = [3\vec{c}_1 + 5\vec{c}_2]_\mathcal{C} = \left(\begin{smallmatrix} 3 \\ 5 \end{smallmatrix}\right)$. Similarly, the second column is $[\vec{b}_2]_\mathcal{C} = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. Thus, $P = \left(\begin{smallmatrix} 3 & 1 \\ 5 & 2 \end{smallmatrix}\right)$.
(b) Since $P[\vec{x}]_\mathcal{B} = [\vec{x}]_\mathcal{C}$ for all $\vec{x} \in V$, then since P is invertible, $[\vec{x}]_\mathcal{B} = P^{-1}[\vec{x}]_\mathcal{C}$ for all $\vec{x} \in V$, so then P^{-1} is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$. Thus, $Q = P^{-1} = \left(\begin{smallmatrix} 2 & -1 \\ -5 & 2 \end{smallmatrix}\right)$. Or, we solve for $\vec{c}_1$ and $\vec{c}_2$ in terms of $\vec{b}_1$ and $\vec{b}_2$ and do the same thing as in part (a).
(c) P takes $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates, and Q does the inverse, so $QP = I_2$.

2. (5 points) Suppose $\vec{v} \in \mathbb{R}^4$ and $\vec{w} \in \mathbb{R}^3$, both nonzero, and let $A = \vec{v}\vec{w}^T$.

   (a) What is the size of A? (Give as $n \times m$ for some n and m.)

   (b) What is $\operatorname{rank} A$?

   (c) What is $\dim \operatorname{Nul} A$?

(a) Since $\vec{v}$ is 4×1 and $\vec{w}$ is 3×1, the product is between a 4×1 and a 1×3 matrix, so A is 4×3.
(b) The product gives $A = (w_1\vec{v} \quad w_2\vec{v} \quad w_3\vec{v})$. Since $\vec{w}$ is nonzero, at least one of the three columns of A is nonzero, and the other two columns are multiples of that column. This means A is rank 1. Said another way,

$$\operatorname{rank} A = \dim \operatorname{Col} A = \dim \operatorname{Span}\{w_1\vec{v}, w_2\vec{v}, w_3\vec{v}\} = \dim \operatorname{Span}\{\vec{v}\} = 1$$

(c) The rank theorem gives $\dim \operatorname{Nul} A = 3 - \operatorname{rank} A = 3 - 1 = 2$.

(For fun) What is the dimension of the subspace of polynomials from $\mathbb{P}_2$ which have 1 as a root?

Reason 1. If a polynomial has 1 as a root, by long division it is divisible by $(x-1)$. So, for a $p(x) \in \mathbb{P}_2$ with 1 as a root, $p(x) = (x-1)q(x)$ for some $q(x) \in \mathbb{P}_1$. We can produce such polynomials by letting $q(x)$ be anything, and every choice of q produces a distinct polynomial $(x-1)q(x)$, so the dimension of the subspace is exactly the same as the dimension of $\mathbb{P}_1$, which is 2.
Reason 2. Let $T : \mathbb{P}_2 \to \mathbb{R}$ be defined by $T(p(x)) = p(1)$. Then, the subspace we are wondering about is $\ker T$. The matrix of T relative to the standard basis $\mathcal{B} = \{1, x, x^2\}$ for $\mathbb{R}_2$ is $A = \left(\begin{smallmatrix} 1 & 1 & 1 \end{smallmatrix}\right)$, and $\dim \operatorname{Nul} A = 2$, so $\dim \ker T = 2$.
Reason 3. If $p(x) = a + bx + cx^2$ has 1 as a root, then $0 = p(1) = a + b + c$. Any solution to this homogeneous equation is a polynomial with 1 as a root, and the dimension of this solution space is 2 since the coefficient matrix has 2 free columns.
Reason 4. Let $T : \mathbb{P}_1 \to \mathbb{P}_2$ be defined by $T(p(x)) = (x-1)p(x)$. By long division, any

polynomial which has 1 as a root is an image of T, and also every image of T is evidentally a polynomial with 1 as a root, so $\operatorname{im} T$ is the subspace we are wondering about. T is injective since whenever $T(p(x)) = 0$, then by dividing both sides by $(x-1)$, we get $p(x) = 0$. Thus, $\dim \operatorname{im} T = \dim \mathbb{P}_1 = 2$.