Quiz 6

1. Let $W = \{A \in \mathbb{R}^{2\times 2} : A^T = A\}$ (the set of 2×2 symmetric matrices).

(a) (4 points) Show that W is a subspace of $\mathbb{R}^{2\times 2}$.

(b) (1 point) Find a basis for W.

(a) We just need to show the three properties:

- Since $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)^T = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, W contains the zero matrix (the "zero vector").
- If A, B are two two-by-two matrices with $A^T = A$ and $B^T = B$, then $(A + B)^T = A^T + B^T = A + B$, so $A + B \in W$, too.
- If $c \in \mathbb{R}$ and A is a two-by-two matrix with $A^T = A$, then $(cA)^T = cA^T = cA$, so $cA \in W$, too.

Alternatively, W is the kernel of $T(A) = A^T - A$, and kernels are subspaces.

(b) $A^T = A$ is just an equation, which we can understand by looking at the entries of A. Suppose $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then $A^T = A$ means $\left(\begin{smallmatrix} a & d \\ b & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, which only implies $b = c$. Thus, vectors of W are just matrices $\left(\begin{smallmatrix} a & b \\ b & d \end{smallmatrix}\right)$ for any a, b, d. This can be rewritten as

$$a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

These three matrices are a spanning set for W. They are independent because the only way to write the zero matrix as a linear combination of them is to take zero of each. Thus, a basis is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

2. (5 points) For the following set W, (a) find a matrix A so that $W = \operatorname{Col} A$, (b) explain why W is a subspace, and (c) give its dimension.

$$W = \left\{ \begin{pmatrix} a + 2b + 4c \\ b + 3c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$

(a) We can just manipulate the definition of W until it looks like $\operatorname{Col} A$ for some A.

$$\begin{aligned} \left\{ \begin{pmatrix} a + 2b + 4c \\ b + 3c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} &= \left\{ a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c \begin{pmatrix} 4 \\ 3 \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} : a, b, c \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix} \vec{x} : \vec{x} \in \mathbb{R}^3 \right\} = \operatorname{Col} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 3 \end{pmatrix} \end{aligned}$$

(b) Column space of a matrix is always a subspace, and W is a column space. Alternatively, the matrix we found has a pivot in every now, so every vector of $\mathbb{R}^2$ is in W, and $\mathbb{R}^2$ is a

subspace.
(c) Since $W = \operatorname{Col} A$, $\dim W = \dim \operatorname{Col} A$. The number of pivots in A is 2, so $\dim W = 2$.

(For fun) A square matrix A which satisfies $A^T = A$ is called *symmetric*, and a square matrix A which satisfies $A^T = -A$ is called *antisymmetric*. Give a way to write any given square matrix as a sum of a symmetric and an antisymmetric matrix. Is this representation unique?

Basically, the average of a matrix and its transpose is a symmetric matrix, and if you subtract that average from the original matrix, you get an antisymmetric matrix. The sum of these two things is the original matrix.
Let $T(A) = \frac{1}{2}(A+A^T)$. The image is symmetric since $T(A)^T = \frac{1}{2}(A^T + (A^T)^T) = \frac{1}{2}(A^T + A) = T(A)$. The image of a symmetric matrix is itself, since if $A = A^T$, then $T(A) = \frac{1}{2}(A+A) = A$, so $T(T(A)) = T(A)$. The matrix $A - T(A)$ is antisymmetric since $(A-T(A))^T = A^T - \frac{1}{2}(A+A^T) = -A + \frac{1}{2}(A + A^T) = -(A - T(A))$. And, $T(A) + (A - T(A)) = A$, which means A is the sum of a symmetric and antisymmetric matrix.
Here is a reason it is a unique representation. If a matrix has two representations, then by subtracting them we get some representation of the zero matrix $0 = A + B$, with A symmetric and B antisymmetric. Since 0 is symmetric, $0 = 0^T = A^T + B^T = A - B$, so $A = B$. But, $0 = A + B$ means $A = -B$, so $A = 0$ and $B = 0$.
Another way to look at all of this is that the set of $n \times n$ matrices is the *direct sum* of two subspaces, the subspace of symmetric matrices and the subspace of antisymmetric matrices. The only matrix which is both symmetric and antisymmetric is the zero matrix, so any matrix which is the sum of a symmetric and an antisymmetric matrix can be represented as such in exactly one way. Also, the dimension of symmetric is equal to the number of entries on or above the diagonal (since the entries below the diagonal are determined by those entries) and the dimension of antisymmetric matrices is equal to th enumber of entries strictly above the diagonal (for the same reason, and because $A^T = -A$ implies the diagonal entries are all zero). The sum of these dimensions is n^2, which is the dimension of $\mathbb{R}^{n \times n}$, they together span all of $\mathbb{R}^{n \times n}$.