Quiz 5

1. (5 points) Suppose A is a square matrix.

   (a) If $\det(A^4) = 0$, explain why A cannot be invertible.

   (b) If A is invertible, explain why $\det(A^T A) > 0$.

(a) Using the product formula $\det(AB) = \det(A)\det(B)$, we have $\det(A^4) = \det(A)^4$, so if $\det(A^4) = 0$, then $\det(A)^4 = 0$, so $\det(A) = 0$, and therefore A is not invertible.
(b) Using the product and transposition formulas for determinant, $\det(A^T A) = \det(A^T)\det(A) = \det(A)\det(A) = \det(A)^2$. If A is invertible, $\det(A) \neq 0$, so $\det(A)^2 > 0$.

2. (5 points) Let $A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

   (a) Compute $\det A$.

   (b) Compute the entry in row 2 column 3 of A^{-1} using the adjugate of A (or Cramer's rule).

(a) Expanding along the third row, the determinant is $-1(1 \cdot 1 - 3 \cdot 2) = 5$.
(b) The $(2, 3)$ entry of $\operatorname{adj}(A)$ is the determinant of the $(3, 2)$ minor times $(-1)^{2+3} = -1$. This is $-1(1 \cdot 1 - 3 \cdot 2) = 5$. Since $\operatorname{adj}(A)/\det(A) = A^{-1}$, then the $(2, 3)$ entry of A^{-1} is $5/5 = 1$.

(For fun) What is the determinant of the following matrix (called a *Vandermonde* matrix)? You should be able to factorize it into a product of three very simple terms.

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}$$

Using the determinant, what are the conditions on a, b, c for the matrix to be invertible?

One could expand along the first column, but I prefer to do some row operations first:

$$\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \sim \begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}$$

These were only replacement row operations, so the determinants of the two matrices are the same. Expanding along the first column, the determinant is $(b-a)(c^2-a^2)-(b^2-a^2)(c-a) = (b-a)(c-a)(c+a)-(b-a)(b+a)(c-a) = (b-a)(c-a)((c+a)-(b+a)) = (b-a)(c-a)(c-b)$. (Notice that if the variables are in the order a, b, c, then each term involves a pair of variables where the first variable comes later in the ordering.)
The only way for the matrix to be non-invertible is for the determinant to be zero, which means either $b = a$, $c = a$, or $c = b$. Therefore the matrix is invertible when $b \neq a$, $c \neq a$, and $c \neq b$. That is, when a, b, c are distinct.