Quiz 4

1. (3 points) For each statement following each matrix shape, circle YES if there is a matrix with that shape where the statement is true, and NO if there is no such matrix.

(a) For a 3×3 matrix:

- (i) The columns might span $\mathbb{R}^3$: $\boxed{\text{YES}}$ (identity matrix)
- (ii) The columns might not span $\mathbb{R}^3$: $\boxed{\text{YES}}$ (zero matrix)
- (iii) The columns might be independent: $\boxed{\text{YES}}$ (identity matrix)
- (iv) The columns might not be independent: $\boxed{\text{YES}}$ (zero matrix)

(b) For a 3×4 matrix:

- (i) The columns might span $\mathbb{R}^3$: $\boxed{\text{YES}}$ (first three rows of I_4)
- (ii) The columns might not span $\mathbb{R}^3$: $\boxed{\text{YES}}$ (zero matrix)
- (iii) The columns might be independent: $\boxed{\text{NO}}$ (can't have 4 pivots)
- (iv) The columns might not be independent: $\boxed{\text{YES}}$ (any matrix)

(c) For a 3×2 matrix:

- (i) The columns might span $\mathbb{R}^3$: $\boxed{\text{NO}}$ (can't have 3 pivots)
- (ii) The columns might not span $\mathbb{R}^3$: $\boxed{\text{YES}}$ (any matrix)
- (iii) The columns might be independent: $\boxed{\text{YES}}$ (first two columns of I_3)
- (iv) The columns might not be independent: $\boxed{\text{YES}}$ (zero matrix)

2. (5 points) For each of the following matrix shapes, complete the corresponding table with all possibilities for rank A and $\dim \operatorname{Nul} A$. Cross out impossible lines.

(a) A is 4×4

rank A	$\dim \operatorname{Nul} A$
0	4
1	3
2	2
3	1
4	0

(b) A is 2×4

rank A	$\dim \operatorname{Nul} A$
0	4
1	3
2	2
~~3~~	~~1~~
~~4~~	~~0~~

(c) A is 4×2

rank A	$\dim \operatorname{Nul} A$
0	2
1	1
2	0
~~3~~	~~−1~~
~~4~~	~~−2~~

These use the rank theorem: the number of columns is the sum of the rank and the dimension of the nullspace. We strike out lines for impossible ranks, since in (b) and (c) there cannot be more than two pivots.

3. (2 points) Compute $\det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 3 & 4 & 0 \end{pmatrix}$.

Expanding along row 2, the determinant is $-\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -(1 \cdot 4 - 2 \cdot 3) = 2$.

(For fun) Is the inverse of an invertible upper triangular matrix always upper triangular?

Yes: all the row operations in $[A \mid I_n] \sim [I_n \mid A^{-1}]$ are scalings or replacements going upward, so since I_n is upper triangular (since diagonal matrices are upper triangular), then A^{-1} must be, too.

We can also calculate an inverse formula. For the case of 3×3 unit upper triangular,

$$\left(\begin{array}{ccc|ccc} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -a & -b+ac \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

I didn't do upper unit triangular for any reason other than not having to introduce three more variables and demonstrating that these two are "closed under taking inverses."