Quiz 3

1. (5 points) The transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x_1,x_2,x_3) = (x_2, x_1+x_3, x_2-x_1)$.
   (a) Write T as a matrix transformation.
   (b) Is T injective (one-to-one)?
   (c) Is T surjective (onto)?

The columns of the standard matrix are T evaluated on the standard basis.

$$\begin{aligned} T(\vec{e}_1) &= T(1,0,0) = (0, 1+0, 0-1) = (0,1,-1) \\ T(\vec{e}_2) &= T(0,1,0) = (1, 0+0, 1-0) = (1,0,1) \\ T(\vec{e}_3) &= T(0,0,1) = (0, 0+1, 0-0) = (0,1,0) \end{aligned}$$

Thus,

$$T(\vec{x}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \vec{x}$$

Row reducing this matrix gets $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)$ so the matrix for T has three pivot columns. There is a pivot in every column, so T is injective, and there is a pivot in every row, so T is surjective. In fact, $\left(\begin{smallmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{smallmatrix}\right)$ is the inverse matrix of the standard matrix of T, so T is invertible, which implies T is both injective and surjective.

2. (5 points)
   (a) Compute the inverse A^{-1} for $A = \left(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 8 \end{smallmatrix}\right)$.
   (b) Solve $AX = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 5 \end{smallmatrix}\right)$ for the 3×3 matrix X.

We compute the inverse by using $[A \mid I_3] \sim [I_3 \mid A^{-1}]$.

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 8 & 0 & 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 7 & -1 & 0 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 1 & 0 & -1 & 3 & -1 \\ 0 & 1 & 0 & -5 & 7 & -2 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix} \\ &\sim \begin{pmatrix} 1 & 0 & 0 & 4 & -4 & 1 \\ 0 & 1 & 0 & -5 & 7 & -2 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix} \end{aligned}$$

So we now know $A^{-1} = \begin{pmatrix} 4 & -4 & 1 \\ -5 & 7 & -2 \\ 2 & -3 & 1 \end{pmatrix}$. (Multiplying $A^{-1}A$ we can check that it is indeed I_3.)
To solve the equation, we left multiply each side by A^{-1} since $A^{-1}AX = X$ to get

$$X = \begin{pmatrix} 4 & -4 & 1 \\ -5 & 7 & -2 \\ 2 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 5 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

Alternatively, one could solve this by noting $AX = [A\vec{x}_1\ A\vec{x}_2\ A\vec{x}_3]$. and then trying to find the columns one at a time. For instance, $A\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is easily solved with $\vec{x}_1 = \vec{e}_1$, since the first column of A is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(For fun) For an $n \times m$ matrix A and an $m \times n$ matrix B, show that if the columns of A do not span $\mathbb{R}^n$, then neither do the columns of AB. (But, if the the columns of B span $\mathbb{R}^m$, then the span of the columns of A equals the span of the columns of AB.)

Let's prove this by the contrapositive. We will show that if the columns of AB do span $\mathbb{R}^n$, then so do the columns of A. Let $\vec{b} \in \mathbb{R}^n$ be an arbitrary vector, and then we will show that $A\vec{x} = \vec{b}$ has a solution (which would imply that the columns of A span $\mathbb{R}^n$). Since the columns of AB span $\mathbb{R}^n$, let $\vec{v}$ be a solution to $AB\vec{v} = \vec{b}$. Then $\vec{x} = B\vec{v}$ is a solution to $A\vec{x} = \vec{b}$.
(So, if the columns of A don't span $\mathbb{R}^n$, neither do the columns of AB.)
Using more recent notation, $\operatorname{Col} AB \subset \operatorname{Col} A$. This means if $\operatorname{Col} A \neq \mathbb{R}^n$, then $\operatorname{Col} AB \neq \mathbb{R}^n$ either.
If the columns of B span $\mathbb{R}^m$, then suppose $\vec{b}$ is a vector where $A\vec{x} = \vec{b}$ has a solution. Then, $B\vec{v} = \vec{x}$ has a solution as well, which means $AB\vec{v} = \vec{b}$. Since we showed every vector in $\operatorname{Col} A$ is a vector in $\operatorname{Col} AB$, then if the columns of B span $\mathbb{R}^m$, then $\operatorname{Col} AB = \operatorname{Col} A$.

(For fun) If A is $n \times n$ with $A^T A = I_n$, how can you solve $A\vec{x} = \vec{b}$ without computing A^{-1}?

By left-multiplying both sides by A^T, since then $A^T A\vec{x} = \vec{x}$, which means $\vec{b} = A^T\vec{b}$. In other words, since A is square and $A^T A = I_n$, A^T just *is* A^{-1}.
Note that if A is not square and $A^T A = I$, then we would again get $\vec{x} = A^T\vec{b}$, but beware: this is only saying that if $A\vec{x} = \vec{b}$ had a solution, then it must be $A^T\vec{b}$. The system might not have any solution at all!