Quiz 2

1. (5 points) Consider the matrix $A = \left(\begin{smallmatrix} 2 & 6 & -4 & -12 \\ -1 & -3 & 4 & 14 \end{smallmatrix}\right)$.
   (a) Solve $A\vec{x} = \vec{0}$ in parametric vector form.
   (b) Give the solution set of $A\vec{x} = \vec{0}$ as the span of some vectors.
   (c) Give a linear dependence relation for the columns of A.

We calculate the reduced row echelon form:

$$\begin{aligned}(A \mid \vec{0}) &= \left(\begin{array}{cccc|c} 2 & 6 & -4 & -12 & 0 \\ -1 & -3 & 4 & 14 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & -6 & 0 \\ -1 & -3 & 4 & 14 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 3 & -2 & -6 & 0 \\ 0 & 0 & 2 & 8 & 0 \end{array}\right) \\ &\sim \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right)\end{aligned}$$

(Note: the augmentation was and always is unnecessary for $A\vec{x} = \vec{0}$. I'm just writing it this way so I don't have to convince anyone of this.)
Thus, x_2 and x_4 are free, $x_1 = -3x_2 - 2x_4$, and $x_3 = -4x_4$. We may then write the parametric form of the solution.

$$\vec{x} = \begin{pmatrix} -3x_2 - 2x_4 \\ x_2 \\ -4x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix}$$

(the intermediate step is unnecssary).
Then, the solution set is

$$\text{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ -4 \\ 1 \end{pmatrix} \right\}$$

Finally, a linear dependence relation is a nontrivial lineal combination of the columns which equals the zero vector. There are many things we may do, but all of them are equivalent to choosing some values for the free variables, at least one of which is nonzero. Let's have $x_2 = 1$ and $x_4 = 0$ for simplicity. This gives the first vector of the span as a solution to $A\vec{x} = \vec{0}$, which written as a linear combination is

$$-3\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 6 \\ -3 \end{pmatrix} + 0\begin{pmatrix} -4 \\ 4 \end{pmatrix} + 0\begin{pmatrix} -12 \\ 14 \end{pmatrix} = \vec{0}$$

2. (5 points) For which values of h are the following column vectors linearly independent?

$$\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\9\\h\end{pmatrix}$$

Let us row reduce the 3×3 matrix formed by these vectors, since a pivot in every column is equivalent means the vectors are linearly independent.

$$\begin{pmatrix}1 & 4 & 7\\2 & 5 & 9\\3 & 6 & h\end{pmatrix} \sim \begin{pmatrix}1 & 4 & 7\\0 & -3 & -5\\0 & -6 & h-21\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 4 & 7\\0 & -3 & -5\\0 & -6 & h-21\end{pmatrix}$$
$$\sim \begin{pmatrix}1 & 4 & 7\\0 & -3 & -5\\0 & 0 & h-11\end{pmatrix}$$

When $h = 11$, the matrix has only two pivots, and otherwise it has three. Therefore, the three vectors are linearly independent when $h \neq 11$.
(As a side note: I mistyped the 9 and meant 8! I wasn't trying to trick anyone!)

(For fun) Consider the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = \vec{x} + \vec{b}$ with $\vec{b} \in \mathbb{R}^3$ a non-zero vector. This transformation represents translation (a shift of space) in the $\vec{b}$ direction. Show that it is not a *linear* transformation. (As in, demonstrate a violation of at least one of the linearity properties.)

There are many reasons. One that illustrates a useful property of linear transformations is this: $T(\vec{0}) = \vec{0} + \vec{b} = \vec{b} \neq \vec{0}$. However, for any linear transformation, $T(\vec{0}) = T(0\vec{0}) = 0T(\vec{0}) = \vec{0}$, using linearity properties (or $T(\vec{0}) = T(\vec{0}+\vec{0}) = T(\vec{0}) + T(\vec{0})$, so by subtracting $T(\vec{0})$ from both sides one gets $T(\vec{0}) = \vec{0}$).
SO: don't try to make a matrix which translates space!
Though, there is something called homogeneous coordinates, which we will not study in Math 54. A three-dimensional vector (x, y, z) can be written in $\mathbb{R}^4$ as $(x, y, z, 1)$, though there is no sense in adding vectors together of this form. Then,

$$\begin{pmatrix}1 & 0 & 0 & a\\0 & 1 & 0 & b\\0 & 0 & 1 & c\\0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\\z\\1\end{pmatrix} = \begin{pmatrix}x+a\\y+b\\z+c\\1\end{pmatrix}$$

which essentially has the effect of translation. This is particularly useful in 3D graphics programming.