Quiz 1

1. (5 points) Consider the augmented matrix $\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 2 & 4 & -1 & 2 & -5 & 0 \\ -1 & -2 & 0 & -3 & 0 & 0 \end{array}\right)$. Give:

   (a) the number of pivot columns for the matrix,
   (b) the number of free variables for the corresponding linear system, and
   (c) the solution to the corresponding linear system.

For each of these, we need the row echelon form of the matrix. Might as well compute the reduced row echelon form so we can read off the solution to the corresponding system easier. First step, $R_2 - 2R_1 \to R_2$ and $R_3 + R_1 \to R_3$. (We may do these simultaneously because no matter the order we perform them we get the same result.)

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 2 & 4 & -1 & 2 & -5 & 0 \\ -1 & -2 & 0 & -3 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & -1 & -4 & -5 & 0 \end{array}\right)$$

Second step, $R_3 + R_2 \to R_3$.

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & -1 & -1 & -5 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Third step, $R_1 + R_2 \to R_1$.

$$\sim \left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From this, we know that (a) there are two pivot columns and (b) three free variables, x_2, x_4, x_5. For (c), we have

$$\begin{aligned} x_1 &= -2x_2 - 3x_4 \\ x_2 &\text{ free} \\ x_3 &= -4x_4 - 5x_5 \\ x_4 &\text{ free} \\ x_5 &\text{ free} \end{aligned}$$

In parametric vector form, this is

$$\vec{x} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}.$$

2. (5 points) Do there exist values for x_1, x_2, and x_3 which solve the following equation?

$$x_1 \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$$

This is just a question about whether the given linear system has a solution. (Using more recent terminology, it is asking whether $(3, 1, 6)$ is a linear combination of the other three vectors.) Since it is just a question of existence, all we need to check is whether the augmented column is a pivot column.

$$\left(\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 1 & 1 & 5 & 1 \\ 2 & -1 & 1 & 6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & 2 & 6 & -2 \\ 0 & 1 & 3 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 1 & 3 & 0 \end{array}\right)$$
$$\sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 3 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

There is a pivot in the last column. Therefore, there is no solution — there do not exist x_1, x_2, x_3 which solve the given equation.

(For fun) For some 3×5 matrix A, the solution set to $A\vec{x} = \vec{0}$ is Span$\{\vec{u}, \vec{v}\}$, with $\vec{u}, \vec{v} \in \mathbb{R}^5$. Do the columns of A span $\mathbb{R}^3$?

Since the solution set is a span of two vectors, $A\vec{x} = \vec{0}$ has at most two free variables. Two free variables means at least three pivots. A has three rows, so A has exactly three pivots, one per row. By a theorem from the book, a pivot in every row means the columns of A span $\mathbb{R}^3$.
Why does span of two vectors mean at most two free variables? This is easiest to understand by the idea of "dimension of a subspace," but it can be answered without the idea, but somewhat more complicatedly. A sketch: when writing the solution to $A\vec{x} = \vec{0}$ in parametric vector form, we get a set of k linearly independent vectors. Writing each of these vectors as a linear combination of $\vec{u}$ and $\vec{v}$, and collecting the weights, we can make a $2 \times k$ matrix B. If $k > 2$, then there is a non-trivial solution $\vec{c}$ to $B\vec{c} = \vec{0}$, and from this we can show the linearly independent vectors are dependent, since $\vec{c}$ itself is a vector of weights which makes the dependence on the vectors.
But, don't worry, this argument will become much easier when we get all the right words to describe it properly.