Quiz 13

1. (5 points) Let $A = \begin{pmatrix} 8 & -9 \\ 6 & -7 \end{pmatrix}$. (a) Give the general solution for $\vec{x}'(t) = A\vec{x}(t)$. (b) Solve the initial value problem for $\vec{x}(0) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

(a)

1. Compute eigenvalues. $\det(A - \lambda i) = (8-\lambda)(-7-\lambda) = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$, so $\lambda = 2, -1$.
2. Compute $\lambda = 2$ eigenvector. $\text{Nul}\begin{pmatrix} 6 & -9 \\ 6 & -9 \end{pmatrix} = \text{Nul}\begin{pmatrix} 2 & -3 \\ 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right\}$.
3. Compute $\lambda = -1$ eigenvector. $\text{Nul}\begin{pmatrix} 9 & -9 \\ 6 & -6 \end{pmatrix} = \text{Nul}\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$.
4. The general solution is

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) Evaluate at $t = 0$ for initial value:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \vec{x}(0) = \vec{x}(t) \qquad = c_1\begin{pmatrix} 3 \\ 2 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

We solve the system to get $c_1 = 1$ and $c_2 = -2$ so the solution is

$$\vec{x}(t) = e^{2t}\begin{pmatrix} 3 \\ 2 \end{pmatrix} - 2e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2. (5 points) The heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = u(1,t) = 0$ and $u(x,0) = f(x)$ for $0 < x < 1$ has the solution $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi)^2 t}\sin(n\pi x)$ for c_n's satisfying $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$. Give the solution for when $f(x) = 2\sin(\pi x) - 5\sin(3\pi x) + 8\sin(7\pi x)$.

The terms of $f(x)$ are $c_1 = 2$, $c_3 = -5$, $c_7 = 8$, and all other c_n's are 0. Thus, the solution is

$$u(x,t) = 2e^{-\pi^2 t}\sin(\pi x) - 5e^{-9\pi^2 t}\sin(3\pi x) + 8e^{-49\pi^2 t}\sin(7\pi x).$$

(For fun) Old console games frequently used *square waves* for their soundtracks. For instance, let $f(x)$ be a periodic function with period 2π which we define on $[-\pi, \pi]$: let $f(x) = -1$ if $-\pi < x < 0$ and $f(x) = 1$ if $0 < x < \pi$. Use the integral inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,dx$ to write $f(x)$ as a linear combination $f(x) = \sum_{n=1}^{\infty} c_n \sin(nx)$. (This is a *Fourier series.* For sound, each term is called a *harmonic.*)

To write $f(x)$ as a linear combination, we use the fact that $\{\sin(nx)\}_n$ is an orthogonal basis with respect to that inner product. Thus, if $f(x)$ is written as $\sum_{n=1}^{\infty} c_n \sin(nx)$, then $\langle f(x), \sin(mx)\rangle = \sum_{n=1}^{\infty} c_n \langle \sin(mx), \sin(nx)\rangle = c_m \langle \sin(mx), \sin(mx)\rangle$, so

$$c_m = \frac{\langle f(x), \sin(mx)\rangle}{\langle \sin(mx), \sin(mx)\rangle}$$

as usual for projection formulae. We calculate

$$\begin{aligned} c_n &= \frac{\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx}{\int_{-\pi}^{\pi} \sin(nx)^2\,dx} \\ &= \frac{\int_{-\pi}^{0} -\sin(nx)\,dx + \int_{0}^{\pi} \sin(nx)\,dx}{\int_{-\pi}^{\pi} \sin(nx)^2\,dx} \end{aligned}$$

which according to Wolfram Alpha is

$$= \frac{2(1-\cos(\pi n))}{n\left(\pi - \frac{\sin(2\pi n)}{2n}\right)}$$

For $n = 1, 2, 3, \ldots$, $\sin(2\pi n) = 0$, and $\cos(\pi n)$ is 1 when n is even and -1 when n is odd, which can be written as $\cos(\pi n) = (-1)^n$. Thus,

$$c_n = \frac{2(1-(-1)^n)}{n}$$

So,

$$f(x) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n} \sin(nx)$$

which, when looked at carefully only has nonzero terms for odd n, so

$$= \sum_{k=1}^{\infty} \frac{4}{2k-1} \sin((2k-1)x)$$

One cool thing about Fourier series is that $\sum_{n=1}^{\infty} c_n^2$ represents the *power* of a signal. In this case, Wolfram Alpha tells me the power is $2\pi^2$. (Though this number doesn't mean that much without comparison to other signals.) Another thing that we can see is that the coefficients decrease only according to the inverse of n, which means it needs quite a lot of sin terms to converge to within a desired error.