Quiz 12

1. (5 points) Consider the differential equation $y'' + y = \sin t$. (a) Find the general solution. (b) Solve the boundary value problem $y(0) = 0$, $y(\pi) = 0$.

(a) The left-hand side has the auxiliary polynomial $r^2 + 1$, which has roots $\pm i$, and the right-hand side corresponds to the roots $\pm i$, so it is virtually a double root for the purpose of the method of undetermined coefficients. The form of a particular solution is $y_p = a_1 t \cos t + a_2 t \sin t$. The derivatives are

$$\begin{aligned} y_p' &= a_1 \cos t - a_1 t \sin t + a_2 \sin t + a_2 t \cos t \\ y_p'' &= -a_1 \sin t - a_1 \sin t - a_1 t \cos t + a_2 \cos t + a_2 \cos t - a_2 t \sin t \end{aligned}$$

Substituting these into $y'' + y$, we get

$$y_p'' + y_p = -2a_1 \sin t + 2a_2 \cos t$$

For this to equal $\sin t$, $a_1 = -\frac{1}{2}$ and $a_2 = 0$, so $y_p = -\frac{1}{2} t \cos t$. The general solution is thus $y(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2} t \cos t$.
(b) The boundary value problem has

$$\begin{aligned} 0 = y(0) &= c_1 + 0 - 0 \\ 0 = y(\pi) &= -c_1 + 0 + \frac{1}{2}\pi \end{aligned}$$

but these equations cannot be simultaneously satisfied, thus the boundary value problem is unsatisfiable. (Note: it was intended for the problem to read $y(\pi/2) = 0$ instead of $y(\pi) = 0$, which you may want to try, but having an unsatisfiable boundary value problem is an OK situation.)

2. (5 points) Verify that $\{\cos^2 x, \sin^2 x, \sin x \cos x\}$ is a fundamental solution set for $y''' + 4y' = 0$.

First of all, we check that they are indeed solutions. While we may substitute them into the differential equation, that is a lot of work. Instead we may recall some trigonometric identities:

$$\begin{aligned} \cos^2 x &= \frac{1}{2} + \frac{1}{2}\cos(2x) \\ \sin^2 x &= \frac{1}{2} - \frac{1}{2}\cos(2x) \\ \sin x \cos x &= \frac{1}{2}\sin(2x) \end{aligned}$$

These are linear combinations of 1, $\cos(2x)$, and $\sin(2x)$, and together correspond to the roots $0, \pm 2i$. An auxiliary polynomial with these three roots is $(r-0)(r-2i)(r+2i) = r^3 + 4r$, so a differential equations they are a solution to is $y''' + 4y = 0$.
Next, we demonstrate that they are linearly independent. Consider the linear transformation

$T(f) = \begin{pmatrix} f(0) \\ f(\pi/4) \\ f(\pi/2) \end{pmatrix}$. Then,

$$T(\cos^2 x) = \begin{pmatrix} 1 \\ 1/2 \\ 0 \end{pmatrix}$$
$$T(\sin^2 x) = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$$
$$T(\sin x \cos x) = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}$$

These three images are linearly independent (three pivots after a quick calculation), so $\cos^2 x$, $\sin^2 x$, and $\sin x \cos x$ are linearly independent.
Or, in the basis $\mathcal{B} = \{1, \cos(2x), \sin(2x)\}$, we have

$$[\cos^2 x]_{\mathcal{B}} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$
$$[\sin^2 x]_{\mathcal{B}} = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$$
$$[\sin x \cos x]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}$$

using the trigonometric identities, and these three coordinate vectors are linearly independent (three pivots, or the 3×3 matrix's determinant is nonzero).
Or, we may calculate the Wronskian $W[\cos^2 x, \sin^2 x, \sin x \cos x](t)$ and show that it is nonzero for some t, but that is quite a lot of calculation.

(For fun) For the mass-spring system governed by $y'' + 2ay' + y = \sin(\omega t)$, $a > 0$, find the amplitude of the oscillation for varying $\omega > 0$. How does this differ from the undampened $a = 0$ case?

(It should say $0 < a < 1$.) The auxiliary polynomial is $r^2 + 2ar + 1$, which has roots $-a \pm i\sqrt{1 - a^2}$. The $\sin(\omega t)$ corresponds to the roots $\pm \omega i$, which is never $-a \pm i\sqrt{1 - a^2}$ since the real part $-a$ is never zero. Thus, $y_p = d_1 \cos(\omega t) + d_2 \sin(\omega t)$. Substituting this into the equation,

$$\begin{aligned} y_p'' + 2ay_p' + y_p =& (-d_1 \omega^2 \cos(\omega t) - d_2 \omega^2 \sin(\omega t)) \\ &+ 2a(-d_1 \omega \sin(\omega t) + d_2 \omega \cos(\omega t)) \\ &+ (d_1 \cos(\omega t) + d_2 \sin(\omega t)) \end{aligned}$$

which, for it to equal $\sin(\omega t)$, gives the following system:

$$\begin{aligned}0 &= -d_1\omega^2 + 2ad_2\omega + d_1 = (1-\omega^2)d_1 + 2a\omega d_2\\ 1 &= -d_2\omega^2 - 2ad_1\omega + d_2 = -2a\omega d_1 + (1-\omega^2)d_2\end{aligned}$$

We then get $d_1 = -2a\omega/((1-\omega^2)^2 + 4a^2\omega^2)$ and $d_2 = (1-\omega^2)/((1-\omega^2)^2 + 4a^2\omega^2)$.
The amplitude of the particular solution is $\sqrt{d_1^2 + d_2^2} = 1/\sqrt{(1-\omega^2)^2 + 4a\omega^2} = 1/\sqrt{1 + 2(2a-1)\omega^2 + \omega^4}$.
To graph this, let us find the local extrema. These occur when the derivative $\frac{d}{d\omega}(1 + 2(2a-1)\omega^2 + \omega^4)$ is 0, which is when $4(2a-1)\omega + 4\omega^3 = 0$, which is either when $\omega = 0$ or $2a-1+\omega^2 = 0$, which is when $\omega = \pm\sqrt{1-2a}$. If $a > \frac{1}{2}$, this never happens, but if $a < \frac{1}{2}$, then it does.
Notice that when $a \to 0$, then this critical point $\sqrt{1-2a}$ approaches 1. When $a = 0$, we have seen how resonance occurs and the double root makes unbounded oscillation with $\omega = 1$, but with dampening, the oscilation is bounded. We can substitute in $\sqrt{1-2a}$ into the amplitude expression to see the maximum amplitude:

$$\begin{aligned}1/\sqrt{1 + 2(2a-1)\omega^2 + \omega^4} &= 1/\sqrt{1 + 2(2a-1)(1-2a) + (1-2a)^2}\\ &= 1/\sqrt{1 - 2(1-2a)^2 + (1-2a)^2}\\ &= 1/\sqrt{1 - (1-2a)^2}\\ &= 1/\sqrt{4a(1-a)}\end{aligned}$$

which, as $a \to 0$ goes to ∞, and at $a = \frac{1}{2}$ is 1.
Interestingly, for $0 < a < \frac{1}{2}$, $1/\sqrt{4a(1-a)} > 1$, which means the system is an *amplifier*.