Quiz 11

1. (5 points) The 2×2 matrix A is symmetric with two eigenvalues 5 and -10. One eigenvector of A is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ corresponding to the eigenvalue -10. Compute A.

Using the spectral theorem for symmetric matrices, the two eigenspaces are orthogonal, so the second eigenvector must be in the span of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, which is a vector orthogonal to $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. This means we have orthogonally diagonalized A, and we may just multiply all the matrices together:

$$\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} -10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -10 & 10 \\ 20 & 5 \end{pmatrix} \begin{pmatrix} 1/5 & -2/5 \\ 2/5 & 1/5 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}. \end{aligned}$$

We may verify that $A \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -10 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 5 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and that A is symmetric.

2. (5 points) Consider $y'' - y' - 6y = 0$. (a) Give the general solution for $y(t)$. (b) Solve the initial value problem for $y(0) = 5$, $y'(0) = 0$.

(a) The auxiliary polynomial is $r^2 - r - 6 = (r-3)(r+2)$, so the roots are $-2, 3$. Thus, the general solution is $y(t) = c_1 e^{-2t} + c_2 e^{3t}$.
(b) The derivative of the general solution is $y'(t) = -2c_1 e^{-2t} + 3c_2 e^{3t}$, so the initial value problem gives the linear system

$$\begin{aligned} 5 &= y(0) = c_1 + c_2 \\ 0 &= y'(0) = -2c_1 + 3c_2 \end{aligned}$$

Using an inverse matrix, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, and thus the solution to the initial value problem is $y(t) = 3e^{-2t} + 2e^{3t}$.

(For fun) Given a homogeneous linear differential equation with constant coefficients, what can you say about $\lim_{t\to\infty} y(t)$ for a solution $y(t)$? Think about the different cases of roots of an auxiliary polynomial, both real and non-real.

When λ is real, $e^{\lambda t}$ approaches 0 if $\lambda < 0$, 1 if $\lambda = 0$, and ∞ if $\lambda > 0$. When $\lambda = \alpha + \beta i$, then $e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$ approaches 0 if $\alpha < 0$, doesn't converge to anything if $\alpha = 0$, and oscillates between ∞ and $-\infty$ if $\alpha > 0$.
When there are multiple roots, then the $\lambda = 0$ case changes and t^n approaches ∞.
When a solution is a linear combination of each of these, then the solution goes to ∞ if any one of them has a real part greater than zero ("unstable"). If all the real parts are less than zero, the solution approaches 0 ("stable").