Quiz 10

1. (5 points) Let W be the span of $\left(\begin{smallmatrix}0\\4\\2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}5\\6\\-7\end{smallmatrix}\right)$. Produce an orthogonal basis for W.

We just need to remove from the second vector the parallel component, according to the Gram-Schmidt process.

$$\frac{(0,4,2)\cdot(5,6,-7)}{(0,4,2)\cdot(0,4,2)}(0,4,2) = \frac{10}{20}(0,4,2) = (0,2,1)$$

Then, $(5,6,-7)-(0,2,1) = (5,4,-8)$ is orthogonal to the first vector. We then have the following orthogonal basis for W:

$$\left\{\begin{pmatrix}0\\4\\2\end{pmatrix}, \begin{pmatrix}5\\4\\-8\end{pmatrix}\right\}$$

2. (5 points) Let V be the span of $\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right)$. Write $T(\vec{x}) = \operatorname{proj}_V \vec{x}$ as a matrix transformation.

The matrix is simply $(T(\vec{e}_1) \quad T(\vec{e}_2))$, which we may compute directly. Alternatively, we may use the projection formula $\operatorname{proj}_V \vec{x} = UU^T\vec{x}$ where U is an orthonormal basis for V. V is one-dimensional, and an orthonormal basis for it is $\left\{\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right\}$, so we compute

$$UU^T = \left(\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}\right)\left(\frac{1}{\sqrt{5}}\begin{pmatrix}2 & 1\end{pmatrix}\right) = \frac{1}{5}\begin{pmatrix}4 & 2\\2 & 1\end{pmatrix}$$

thus

$$T(\vec{x}) = \begin{pmatrix}4/5 & 2/5\\2/5 & 1/5\end{pmatrix}\vec{x}.$$

We check that $T(2,1) = (2,1)$.

(For fun) Consider the transformation $T : V \to V$ defined by $T(f) = f''$, where V is the set of functions whose second derivatives are continuous. What are the eigenvectors of T?

An eigenvector is a nonzero function f where $T(f) = \lambda f$ for some λ. That is, $f'' - \lambda f = 0$.
The auxiliary polynomial is $r^2 - \lambda$, which has roots $\pm\sqrt{\lambda}$.
If $\lambda > 0$, then the corresponding eigenspace is two-dimensional, spanned by $e^{\sqrt{\lambda}t}$ and $e^{-\sqrt{\lambda}t}$.
If $\lambda = 0$, then the double root means the eigenvectors are 1 and t.
If $\lambda < 0$, then the roots are purely imaginary, with eigenvectors $\cos(\sqrt{-\lambda}t)$ and $\sin(\sqrt{-\lambda}t)$.