Discussion – Oct. 12

1. For $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $\{I, A, A^2\}$ is a dependent set (hint: compute $\det(A)I - tr(A)A + A^2$, where $tr(A)=a+d$)

2. (1) Show $\mathcal{B}=\{(x-2)(x-3), (x-1)(x-3), (x-1)(x-2)\}$ is a basis for $\mathbb{P}_2$.
   (2) Let $T: \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ be $T(p(x)) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix}$ (evaluation at $x=1,2,3$).
   What is the matrix of T rel. $\mathcal{B}$ and the std. basis of $\mathbb{R}^3$?
   (3) Is T invertible? (If so, what is a formula for $T^{-1}: \mathbb{R}^3 \to \mathbb{P}_2$?)
   (Ok, it is. The inverse is called Lagrange interpolation)

3. For A $m\times n$, $T: V \to W$ linear ($\dim V = n$, $\dim W = m$), make sense of "Col is to Im as Nul is to Ker."
   If A is the matrix of T rel. bases of V and W, make more sense of it.

4. Compute dimensions:
   (1) $\mathbb{P}_5$ (2) $\{\vec{x}\in\mathbb{R}^4 \mid x_1+x_2+x_3+x_4=0\}$
   (3) Col A when $n\times n$ A is invertible
   (4) Col A when 3×3 A has Nul $A = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$
   (5) im T and ker T for $T: \mathbb{P}_3 \to \mathbb{P}_3$, $T(p(x)) = p(x) - x\,p'(x)$ (derivative)
   (6) $\left\{ A \in \mathbb{R}^{2\times 2} \;\middle|\; \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} A = A \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$
   (7) $\left\{ \vec{v}\in\mathbb{R}^2 \;\middle|\; \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\vec{v} = 3\vec{v} \right\}$ (8) Nul $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \end{pmatrix}$

5. Let $\mathcal{B} = \{\cos x, \sin x\}$. Find $\left[\sin\left(x-\frac{\pi}{4}\right)\right]_{\mathcal{B}}$

6. If V is n-dimensional, what can you say about m vectors if (a) $m<n$ (b) $m>n$ (c) $m=n$?