Discussion comments

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Unlike usual, I'm giving some solutions to provide a sort of model for writing solutions. However, they can still be written better! (I'm on a time budget.)

1. For each of the following sets, determine whether it (1) has the zero vector (2) is closed under addition and (2) is closed under scalar multiplication.

   (a) The set of odd integers (i.e., $\{n \in \mathbb{Z} : n = 2k+1 \text{ for some } k \in \mathbb{Z}\}$). This is a subset of the vector space $\mathbb{R}$, and it is missing 0. The sum of any two odd integers is even, for instance $1+1=2$, and twice any odd integer is even, for instance $2 \cdot 1 = 2$.

   (b) The set of even integers (i.e., $\{n \in \mathbb{Z} : n = 2k \text{ for some } k \in \mathbb{Z}\}$). It has 0. $n_1 + n_2 = 2k_1 + 2k_2 = 2(k_1+k_2)$, so the sum of even integers is even. It is not closed under scalar multiplication: $\frac{1}{2} \cdot 2 = 1$ is not even.

   (c) $\{A \in \mathbb{R}^{2\times 2} : \det A = 1\}$. The zero vector of $\mathbb{R}^{2\times 2}$ is the two-by-two zero matrix, whose determinant is 0, so the set is missing the zero vector. Since $\det I_2 = 1$, the identity matrix is in the set, but $\det(I_2+I_2) = 4$ and $\det(2I_2) = 4$, so it is not closed under either addition or scalar multiplication.

   (d) $\{A \in \mathbb{R}^{2\times 2} : a_{21} = 0\}$. These, by the way, are upper triangular 2×2 matrices. The zero matrix has the $(2,1)$-entry equal to 0, so the set contains the zero vector. Suppose A, B are both in the set. Then the $(2,1)$ entry of $A+B$ is $a_{21} + b_{21} = 0 + 0 = 0$, so it is closed under addition. Finally, cA has entry $(2,1)$ being $ca_{21} = c0 = 0$, so it is closed under scalar multiplication. (Alternatively: this set is the kernel of the transformation $T(A) = a_{21}$, so it is a subspace.)

   (e) $\{A \in \mathbb{R}^{2\times 2} : \text{all entries of } A \text{ are negative}\}$. It is missing the zero matrix since 0 is not negative. Adding two matrices with negative entries results such a matrix, since negative plus negative is negative. Scaling such a matrix by -1 results in a matrix whose entries are all *positive*.

   (f) For $B \in \mathbb{R}^{3\times 2}$ some unknown matrix, $\{A \in \mathbb{R}^{2\times 2} : BA = 0\}$, where 0 represents the zero matrix. The zero matrix has $B0 = 0$, so 0 is such a matrix. Given A_1 and A_2 with $BA_1 = 0$ and $BA_2 = 0$, then $B(A_1 + A_2) = BA_1 + BA_2 = 0 + 0 = 0$, so $A_1 + A_2$ is also in the set. Similarly, $B(cA) = cBA = c0 = 0$, so it is also closed under scalar multiplication. (By the way, all that needs to be true of A is that its columns lie in $\operatorname{Nul} B$. In a more abstract linear algebra course, we would say that the set is the subspace isomorphic to $\operatorname{Nul}(A) \oplus \operatorname{Nul}(A)$.) (Alternatively, this is the kernel of the transformation $T(A) = BA$, which you'd need to check is a linear transformation.)

   (g) $\{A \in \mathbb{R}^{2\times 2} : A^4 = 0\}$. The fourth power of the zero matrix is zero, so the set has the zero vector. It is not closed under addition: it contains both $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, whose sum is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, but the fourth power of this matrix is I_2, not zero. It is closed under scalar multiplication since $(cA)^4 = c^4 A^4 = c^4 0 = 0$.

   (h) $\{p(x) \in \mathbb{P}_2 : p(3) = 0\}$ (that is, at-most-second-degree polynomials which have 3 as a root). The zero polynomial has 3 as a root. If p, q are in the set, then $(p+q)(3) = p(3) + q(3) = 0 + 0 = 0$. Similarly, $(cp)(3) = cp(3) = c0 = 0$. (Alternatively, this is the kernel of the evaluation map $T(p(x)) = p(3)$, which you'd need to check is a linear transformation.) (Alternatively alternatively, having a root means the polynomial is divisible by $(x-3)$, so the set is of all $p(x)$ which can be written as $q(x)(x-3)$, which is the image of $T(q(x)) = q(x)(x-3)$, a linear transformation.)

(i) $\{p(x) \in \mathbb{P}_2 : p(3) = 1\}$. This is missing the zero polynomial since at 3 the zero polynomial is 0, not 1. It is not closed under addition: for any p, q in the set, $(p+q)(3) = p(3)+q(3) = 1+1 = 2$. It is not closed under scalar multiplication: $(cp)(3) = cp(3) = c$, so if $c \neq 1$ it leaves the set.

(j) $\{f(x) \text{ continuous} : f(3) = 0\}$ similar to (h).

(k) $\{f(x) \text{ continuous} : f(3) = 1\}$ similar to (i).

(l) $\{f(x) \text{ differentiable} : f'(3) = 0\}$. The zero function has 0 as its derivative, so it's in the set. Given f, g in the set $(f+g)'(3) = f'(3) + g'(3) = 0 + 0 = 0$, and $(cf)'(3) = cf'(3) = c0 = 0$. (This is using linearity of the transformation $\frac{d}{dx}$.) (This is the kernel of the linear transformation $\frac{d}{dx}\big|_{x=3}$.)

(m) $\{f(x) \text{ differentiable} : f'(x) = 0 \text{ for all } x\}$. This is similar to (l). Another way is to notice that $f'(x) = 0$ means $f(x) = c$ for some constant c (by the mean value theorem), so the set is actually the set of constant functions — it is isomorphic to $\mathbb{R}$.

(n) $\{f(x) : f(x) = f(-x)\}$. For $f(x) = 0$ (the zero function), $f(-x) = 0 = f(x)$, so it is in it. For f, g in the set, $(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x)$, so $f+g$ is in it. Also, $(cf)(-x) = cf(-x) = cf(x) = (cf)(x)$. (This is the kernel of $T(f(x)) = f(x) - f(-x)$.)

(o) $\{f(x) : f(x) = -f(-x)\}$. For f the zero function, $f(-x) = 0 = -0 = -f(x)$. For f, g in the set, $(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f(x) + g(x)) = -(f+g)(x)$. Also, $(cf)(-x) = cf(-x) = -cf(x) = -(cf)(x)$. (This is either the image of the previous transformation or the kernel of $S(f(x)) = f(x) + f(x)$.)

(p) $\{f(x) : \text{for } R \text{ the radius of convergence of } f \text{ at } 0, R > 0\}$. The zero function has a radius of convergence of ∞, so the zero function is in it. If f, g in the set with R_f and R_g the respective radii of convergence, a Math 1B fact is the radius of convergence R_{f+g} of $f+g$ is at least the smaller of R_f and R_g. Also, scaling a function does not change its radius of convergence (unless it is scaling by 0, in which case the radius becomes ∞).