Math 1B: Calculus **Spring 2020**

Discussion 21: Modeling With Differential Equations (9.1)

Instructor: Alexander Paulin Date: Mar 30, 2020

1. Verify that the following differential equations have the given solutions, where y is a function of t.

   (a) For $y' = ay$, the solutions $y = Ce^{at}$ for $C \in \mathbb{R}$.

   (b) For $y'' + y = 0$, the solutions $y = C\sin(t)$ and $y = C\cos(t)$ for $C \in \mathbb{R}$.

   (c) For $ty' = 1$, the solutions $y = \ln(t) + C$ for $C \in \mathbb{R}$.

2. Determine the general solution to the given differential equation.[1]

   (a) $y' = 0$.

   (b) $y'(t) = \sin(t)$

[1] *(Hint: this is integration by another name.)*

(c) $y'(t) = t^2$

(d) $y''(t) = -\frac{1}{t^2}$

3. Find the equilibrium solutions[2] to the following differential equations.

(a) $\frac{dy}{dt} = ay$ for $a \neq 0$. (What happens if $a = 0$?)

(b) $\frac{dy}{dt} = 1 - y^2$.

(c) $\frac{dy}{dt} = 1 + y^2$.

(d) $\frac{dy}{dt} = ay(b - y)$ for $a \neq 0$.

(e) $\frac{dy}{dt} = \cos(y)$.

[2]An *equilibrium solution* is a constant-valued solution $y(t) = c$ for some $c \in \mathbb{R}$.

4. An object of mass m hanging from an ideal spring with spring constant k obeys the second-order differential equation

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x,$$

where $x(t)$ is the vertical displacement through time.

(a) Show that $x(t) = A\cos(t\sqrt{k/m}) + B\sin(t\sqrt{k/m})$ is a solution, where $A, B \in \mathbb{R}$.

(b) Solve the initial value problem where $x(0) = 1$ and $x'(0) = 0$, representing releasing the object from one unit of displacement at rest.

(c) Solve the initial value problem where $x(0) = 0$ and $x'(0) = 1$, representing giving the object a unit-sized kick at time zero.

5. Suppose $g(t)$ is a function. Write the solution $y(t)$ for $y'(t) = g(t)$ using a definite integral, given the initial condition $y(0) = C$.

6. Guess the general solution $y(t)$ for $y^{(n)} = 0$. *(Hint: What kinds of functions become zero after repeated differentiation?)*

7. Suppose $y = f(t)$ and $y = g(t)$ are two solutions to $y'' + ay' + by = 0$. Show that $y = Af(t) + Bg(t)$ is also a solution, for all constants $A, B \in \mathbb{R}$.

8. An equilibrium solution $y(t) = c$ for a first-order differential equation $y'(t) = g(y)$ is called *stable* if there is an interval (a, b) containing c such that $g(y)$ is positive for $y \in (a, c)$ and $g(y)$ is negative for $y \in (c, b)$. Analyze the stability of equilibrium solutions in problem 3. Can you make intuitive sense of this terminology?