Math 1B

Power series representations

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1. (i) Find a power series centered at 0 for the function
 (ii) Determine its interval of convergence
 (iii) Do the same for all possible centers.

$\sum_{n=0}^{\infty} a_n (x-b)^n$

(a) $f(x) = \dfrac{3}{2+x} = \dfrac{3/2}{1-\left(\frac{-x}{2}\right)} = \sum_{n=0}^{\infty} \frac{3}{2}\cdot\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{3}{2}\cdot(-1)^n \cdot \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \underbrace{\frac{3(-1)^n}{2^{n+1}}}_{a_n} x^n \qquad b=0$

$-1 < r < 1$ for convergence

$-1 < \frac{-x}{2} < 1$

$-2 < -x < 2$

$2 > x > -2$ $\qquad$ $(-2,2)$ is interval of convergence

Ratio test:

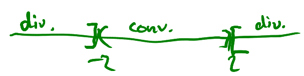
$$\lim_{n\to\infty} \left| \frac{\frac{3\cancel{(-1)^{n+1}}\, x^{n+1}}{2^{n+1}}}{\frac{3\cancel{(-1)^n}\, x^n}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{x}{2} \right| = \frac{|x|}{2}$$

$\frac{|x|}{2} < 1$ converges $\qquad$ $\frac{|x|}{2} > 1$ diverges

$\Leftrightarrow |x| < 2$ $\qquad$ $\Leftrightarrow x < -2$ or $2 < x$

$\Leftrightarrow -2 < x < 2$

@ $x = 2$, $\sum_{n=0}^{\infty} \frac{3(-1)^n}{2} = \frac{3}{2} - \frac{3}{2} + \frac{3}{2} - \cdots$ diverges



@ $x = -2$, $\sum_{n=0}^{\infty} \frac{3}{2} = \frac{3}{2} + \frac{3}{2} + \cdots$ diverges

(b) $f(x) = \dfrac{5}{1-\underbrace{4x^2}_{r}} = \sum_{n=0}^{\infty} 5\,(\overbrace{4x^2}^{4^n (x^2)^n})^n = \sum_{n=0}^{\infty} 5\cdot 4^n \cdot x^{2n}$

$(2n = m) \qquad = \sum_{\substack{m=0 \\ m \text{ even}}}^{\infty} 5\cdot 4^{m/2} x^m$

$-1 < 4x^2 < 1$ (from geometric series)

$\Leftrightarrow 4x^2 < 1$

$\Leftrightarrow x^2 < 1/4 \Leftrightarrow \boxed{-1/2 < x < 1/2}$ interval of convergence

(c) $f(x) = \dfrac{1}{x^2+b^2}$

$-1 < 1 - \frac{t}{5} < 1$

(d) $f(x) = \ln(5+x) = \int_1^{5+x} \frac{1}{t}\,dt = \int_1^{5+x} \frac{1}{5-(5-t)}\,dt = \int_1^{5+x} \frac{1/5}{1-\left(1-\frac{t}{5}\right)}\,dt = \int_1^{5+x} \sum_{n=0}^{\infty} \frac{1}{5}\left(1-\frac{t}{5}\right)^n dt$

$= \sum_{n=0}^{\infty} \int_1^{5+x} \frac{1}{5}\left(1-\frac{t}{5}\right)^n dt = \sum_{n=0}^{\infty} \left[\frac{-1}{n+1}\left(1-\frac{t}{5}\right)^{n+1}\right]_1^{5+x} = \sum_{n=0}^{\infty} \left(\frac{-1}{n+1}\left(\frac{-x}{5}\right)^{n+1} + \frac{1}{n+1}\left(\frac{4}{5}\right)^{n+1} \right)$

$= \sum_{n=0}^{\infty} \frac{-(-1)^{n+1} x^{n+1}}{(n+1)5^{n+1}} + \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\frac{4}{5}\right)^{n+1}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)5^{n+1}} + \ln(5) = \ln(5) - \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{5^n n}$

(e) $f(x) = \ln(5-x)$

(f) $f(x) = \dfrac{2x+3}{x^2+3x+2}$

(g) $f(x) = \dfrac{1+x}{(1-x)^2}$

(h) $f(x) = \tan^{-1}(2x)$