Power series representations

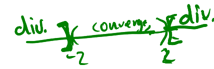
1. (i) Find a power series centered at 0 for the function
   (ii) Determine its interval of convergence
   (iii) Do the same for all possible centers.

(a) $f(x) = \dfrac{3}{2+x} = \dfrac{3/2}{1+\frac{x}{2}} = \dfrac{3/2}{1-\left(\frac{-x}{2}\right)} = \displaystyle\sum_{n=0}^{\infty} \frac{3}{2}\left(\frac{-x}{2}\right)^n = \sum_{n=0}^{\infty} \frac{3(-1)^n x^n}{2\cdot 2^n} = \sum_{n=0}^{\infty} \underbrace{\frac{(-1)^n 3x^n}{2^{n+1}}}_{a_n}$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{3x^{n+1}}{2^{n+2}}}{\frac{3x^n}{2^{n+1}}}\right| = \lim_{n\to\infty}\frac{|x|}{2} = \frac{|x|}{2}$

if $\frac{|x|}{2} < 1$, converges

$\Leftrightarrow |x| < 2$



if $\frac{|x|}{2} > 1$, diverges

$\Leftrightarrow |x| > 2$

$-1 < \frac{-x}{2} < 1$ ← from knowlege of geometric series

$\Leftrightarrow -2 < -x < 2$ ← interval of convergence

$\Leftrightarrow 2 > x > -2$

(b) $f(x) = \dfrac{5}{1-4x^2} = \displaystyle\sum_{n=0}^{\infty} 5\cdot(4x^2)^n = \sum_{n=0}^{\infty} 5\cdot 4^n\cdot x^{2n}$

$-1 < 4x^2 < 1$ ← from knowlege of geometric series

$\Leftrightarrow -1/4 < x^2 < 1/4$

$\Leftrightarrow 0 \leq x^2 < 1/4$

$\Leftrightarrow -1/2 < x < 1/2$ ← interval of convergence

(c) $f(x) = \dfrac{1}{x^2+b^2}$

(d) $f(x) = \ln(5+x)$

(e) $f(x) = \ln(5-x)$

(f) $f(x) = \dfrac{2x+3}{x^2+3x+2} = \dfrac{2x+3}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} = \dfrac{1}{x+1} + \dfrac{1}{x+2}$

$$= \frac{1}{1-(-x)} + \frac{1/2}{1-(-x/2)}$$

$$= \sum_{n=0}^{\infty} (-x)^n + \sum_{n=0}^{\infty} \frac{1}{2}(-x/2)^n$$

$$= \sum_{n=0}^{\infty} \left((-1)^n + \frac{1}{2}\left(-\frac{1}{2}\right)^n\right) x^n$$

(g) $f(x) = \dfrac{1+x}{(1-x)^2}$

(h) $f(x) = \tan^{-1}(2x)$