Math 1B: Calculus	Spring 2020
Discussion 25: Linear Equations	
Instructor: Alexander Paulin	Date: Apr 10, 2020

1. Solve the differential equation:
   a. $4x^3y + x^4y' = \sin^3 x$

1 a) $4x^3 y' + x^4 y = \sin^3 x$

$$y' + \frac{4}{x} y = x^{-4} \sin^3 x \qquad I = e^{\int \frac{4}{x} dx} = e^{4 \ln |x|} = x^4$$

$$(x^4 y)' = \sin^3 x$$

$$x^4 y = \int \sin^3 x \, dx$$

$$x^4 y = \int (1 - \cos^2 x) \sin x \, dx \qquad u = \cos x, \quad du = -\sin x \, dx$$

$$x^4 y = -\int (1 - u^2) \, du$$

$$x^4 y = -u + \frac{u^3}{3} + C$$

$$x^4 y = -\cos x + \frac{\cos^3 x}{3} + C$$

$$\boxed{y = -x^{-4} \cos x + \frac{1}{3} x^{-4} \cos^3 x + C x^{-4}}$$

b. $t^2 \frac{dy}{dt} + 3ty = \sqrt{1+t^2}, t > 0$

1 b) $t^2 \frac{dy}{dt} + 3ty = \sqrt{1+t^2}$ $\qquad I(t) = e^{\int 3/t} = e^{3\ln t} = t^3$

$$\frac{dy}{dt} + \frac{3}{t} y = t^{-2}\sqrt{1+t^2}$$

$$\Rightarrow t^3 \frac{dy}{dt} + 3t^2 y = t\sqrt{1+t^2}$$

$$(t^3 y)' = t\sqrt{1+t^2}$$ $\qquad u = 1+t^2$

$$t^3 y = \int t\sqrt{1+t^2}\, dt$$ $\qquad du = 2t\, dt$

$$t^3 y = \frac{1}{2}\int u^{1/2}\, du$$

$$t^3 y = \frac{1}{2} \frac{u^{3/2}}{3/2} + C$$

$$\boxed{y = \frac{1}{3} t^{-3} (1+t^2)^{3/2} + C t^{-3}}$$

c. $xy' - 2y = x^2, x > 0$

1 c)

$$xy' - 2y = x^2 \qquad x > 0$$

$$I(x) = e^{\int -2/x\,dx} = e^{-2\ln x} = x^{-2}$$

$$y' - \frac{2}{x}y = x$$

$$x^{-2}y' - 2x^{-3}y = x^{-1}$$

$$(x^{-2}y)' = \int x^{-1}$$

$$x^{-2}y = \int x^{-1}\,dx$$

$$x^{-2}y = \ln x + c$$

$$\boxed{y = x^2 \ln x + Cx^2}$$

2. Solve the initial-value problem
   a. $t\frac{du}{dt} = t^2 + 3u, t > 0, u(2) = 4$

2 a) $t\frac{du}{dt} = t^2 + 3u$ $\quad u(2) = 4, \; t > 0$

$I(t) = e^{\int -3/t \, dt} = e^{\ln t^{-3}} = t^{-3}$

$t u' - 3u = t^2$

$u' - \frac{3}{t} u = t$

$t^{-3} u' - 3t^{-4} u = t^{-2}$

$(t^{-3} u)' = t^{-2}$

$t^{-3} u = \int t^{-2} dt = \frac{t^{-1}}{-1} + C$

$u = -t^2 + C t^3$

$u(2) = 4 = -4 + C8$

$C = 1$

$\Rightarrow \boxed{u = -t^2 + t^3}$

b. $xy' + y = xlnx, y(1) = 0$

2 b) $xy' + y = x \ln x \qquad y(1) = 0$

$(xy)' = x \ln x \qquad u = \ln x \quad dv = x\,dx$

$xy = \int x \ln x \, dx \qquad du = \frac{1}{x} dx \quad v = \frac{x^2}{2}$

$xy = \frac{x^2}{2} \ln|x| - \int \frac{1}{2} x \, dx$

$xy = \frac{x^2}{2} \ln(x) - \frac{1}{2} \cdot \frac{x^2}{2} + C$

$y = \frac{x}{2} \ln(x) - \frac{1}{4} x + C x^{-1}$

$y(1) = 0 = 0 - \frac{1}{4} + C \Rightarrow C = \frac{1}{4}$

$$\boxed{y = \frac{x}{2} \ln(x) - \frac{1}{4} x + \frac{1}{4} x^{-1}}$$