Integration by parts

These are solutions and discussion for selected problems.

2. (d) $\int x^n \sin(x)\,dx$ and (e) $\int x^n \cos(x)\,dx$ for integers $n = 0, 1, 2, \ldots$.

Do integration by parts with

$$\begin{aligned} f(x) &= x^n & g'(x) &= \sin(x) \\ f'(x) &= nx^{n-1} & g(x) &= -\cos(x). \end{aligned}$$

Then,

$$\begin{aligned} \int x^n \sin(x)\,dx &= (x^n)(-\cos(x)) - \int (nx^{n-1})(-\cos(x))\,dx \\ &= -x^n \cos(x) + n \int x^{n-1} \cos(x)\,dx. \end{aligned}$$

So, the integral reduces to one of the form for part (e). Integration by parts for (e) gives

$$\int x^n \cos(x)\,dx = x^n \sin(x) - n \int x^{n-1} \sin(x)\,dx.$$

(Check these by taking the derivatives of both sides.)
These reduction rules are enough to compute these integrals for any particular integer value of $n \geq 0$.
One thing you can do with these is substitute the second rule into the first (with the appropriate modification of n):

$$\begin{aligned} \int x^n \sin(x)\,dx &= -x^n \cos(x) + n \left(x^{n-1} \sin(x) - (n-1) \int x^{n-2} \sin(x)\,dx \right) \\ &= -x^n \cos(x) + nx^{n-1} \sin(x) - n(n-1) \int x^{n-2} \sin(x)\,dx. \end{aligned}$$

If you keep expanding it a few times, you might realize that, fully expanded, the integral can be written in the following form:

$$\begin{aligned} \int x^n \sin(x)\,dx = &- \left(x^n - \tfrac{n!}{(n-2)!} x^{n-2} + \tfrac{n!}{(n-4)!} x^{n-4} - \tfrac{n!}{(n-6)!} x^{n-6} + \cdots \right) \cos(x) \\ &+ \left(\tfrac{n!}{(n-1)!} x^{n-1} - \tfrac{n!}{(n-3)!} x^{n-3} + \tfrac{n!}{(n-5)!} x^{n-5} - \cdots \right) \sin(x) + C. \end{aligned}$$

The stuff in the dots keeps going until the x or constant term.

3. $\int x^n e^x\,dx$ for $n = 0, 1, 2, \ldots$.

Integration by parts with

$$f(x) = x^n \qquad g'(x) = e^x$$
$$f'(x) = nx^{n-1} \qquad g(x) = e^x$$

gives

$$\int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx.$$

This is a good enough general solution since it reduces the integral into a simpler one. Again, we can keep substituting the integral into itself, like

$$\int x^n e^x \, dx = x^n e^x - n \left(x^{n-1} e^x - (n-1) \int x^{n-2} e^x \, dx \right) = \cdots ,$$

and then try to figure out the pattern. What I came up with is

$$\int x^n e^x \, dx = \left(x^n - \tfrac{n!}{(n-1)!} x^{n-1} + \tfrac{n!}{(n-2)!} x^{n-2} - \tfrac{n!}{(n-3)!} x^{n-3} + \cdots \right) e^x + C.$$

The last term in the "$\cdots$" is $\pm n!$, with the sign depending on whether n is even or odd.

5. $\int \arctan(x) \, dx$

Since we know the derivative of $\arctan(x)$, let us use the following choice for integration by parts:

$$f(x) = \arctan(x) \qquad g'(x) = 1$$
$$f'(x) = \tfrac{1}{1+x^2} \qquad g(x) = x.$$

Then,

$$\int \arctan(x) \, dx = x \arctan(x) - \int \frac{x}{1+x^2} \, dx$$

With $u = 1 + x^2$, the second integral (with $du = 2x \, dx$) is

$$\int \frac{x}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| = \frac{1}{2} \ln|1 + x^2|.$$

Substituting this back in,

$$\int \arctan(x) \, dx = x \arctan(x) - \frac{1}{2} \ln(1 + x^2) + C.$$

We may omit the absolute value signs since $1 + x^2 > 0$ for all x.