Quiz 5

1. (3 points) Does the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{26-10n+n^2}$ converge or diverge?

Completing the square for the denominator, we have $26 - 10n + n^2 = (n-5)^2 + 1$. Notice that this means the denominator decreases until $n = 5$ and then increases. So, to simplify things, let us consider instead the series $\sum_{n=6}^{\infty} \frac{(-1)^n}{26-10n+n^2}$. This is certainly an alternating series, since the denominator is always positive. And, from having completed the square, we know $\frac{1}{26-10n+n^2}$ is monotonic decreasing. Furthermore, we see $\lim_{n\to\infty} \frac{1}{26-10n+n^2} = 0$ by appealing to our knowledge of Math 1A. Hence, this truncated series satisfies the hypotheses of the Alternating Series Test and thus converges. Truncating a series does not change convergence properties, therefore the given series **converges**.

Alternatively, we may show the series absolutely converges by using the limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{26-10n+n^2}\right)} &= \lim_{n\to\infty} \frac{26-10n+n^2}{n^2} \\ &= \lim_{n\to\infty} (26n^{-2} - 10n^{-1} + 1) \\ &= 1\end{aligned}$$

This is strictly between 0 and ∞, so by the limit comparison test the given series $\sum_{n=1}^{\infty} \frac{1}{26-10n+n^2}$ converges, so the given series **(absolutely) converges**.

2. (3 points) Suppose $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = 2$. For which values of x does the series $\sum_{n=1}^{\infty} \frac{x^n}{b_1 b_2 \cdots b_n}$ converge?

Note: *I made a mistake with this problem. The convergence at $x = \pm 2$ depends very much on what $\{b_n\}$ is, and the hints don't seem to make any sense! Sorry about that!*

Let us start setting up the ratio test, since many things will beneficially cancel. We are considering the limit

$$\lim_{n\to\infty} \left| \frac{\left(\frac{x^{n+1}}{b_1 b_2 \cdots b_n b_{n+1}}\right)}{\left(\frac{x^n}{b_1 b_2 \cdots b_n}\right)} \right| = \lim_{n\to\infty} \left| \frac{x}{b_{n+1}} \right|,$$

and according to the given hypothesis, $\lim_{n\to\infty} b_n = 2$, so we may substitute this into the above limit to simplify this to

$$= \left| \frac{x}{2} \right| = \tfrac{1}{2}|x|.$$

The ratio test says that if $\frac{1}{2}|x| < 1$, then the series converges, and if $\frac{1}{2}|x| > 1$ the series diverges. Thus, we so far have that the series converges if $|x| < 2$ and diverges if $|x| > 2$.

This leaves when $|x| = 2$. However, as I stated, I made a mistake with this problem. Consider the following three b_n sequences:

1. If $b_n = 2$ for all n, then the series is $\sum_{n=1}^{\infty} (\frac{x}{2})^n$, which diverges if $x = \pm 2$.
2. If $b_n = \frac{2(n+1)}{n}$, then

$$\begin{aligned} b_1 b_2 b_3 \dots b_{n-1} b_n &= \tfrac{2\cdot 2}{1} \cdot \tfrac{2\cdot 3}{2} \cdot \tfrac{2\cdot 4}{3} \cdots \tfrac{2n}{n-1} \cdot \tfrac{2(n+1)}{n} \\ &= 2^n (n+1). \end{aligned}$$

   Hence, the series is $\sum_{n=1}^{\infty} \frac{x^n}{2^n(n+1)} = \sum_{n=1}^{\infty} \frac{(x/2)^n}{n+1}$. When $x = 2$, this is a harmonic series so diverges, and when $x = -2$, this is an alternating series that the Alternating Series Test says converges.
3. With $b_n = \frac{2(n+1)^2}{n^2}$, then by similar reasoning the series is $\sum_{n=1}^{\infty} \frac{(x/2)^n}{(n+1)^2}$. When $x = 2$, the series converges because it is a p-series, and when $x = -2$, the series converges because of the Alternating Series Test.

Remember that if a series is absolutely convergent, then it is convergent. This means that, since the series has positive coefficients, if the series at $x = 2$ converges, it converges at $x = -2$ as well. Hence, these three examples show all the possible behavior:

	Converges at	
$\{b_n\}$	$x = 2$	$x = -2$
$b_n = 2$	no	no
$b_n = \frac{2(n+1)}{n}$	no	yes
$b_n = \frac{2(n+1)^2}{n^2}$	yes	yes

That is to say, we cannot conclude anything about convergence at $x = \pm 2$, other than "if it converges at $x = 2$, then it converges at $x = -2$ as well."

3. (3 points) Does the series $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ converge or diverge?[1]

Let's manipulate the denominator:

$$\begin{aligned} \ln(n)^{\ln(n)} &= \left(e^{\ln(\ln(n))}\right)^{\ln(n)} \\ &= e^{\ln(n)\ln(\ln(n))} \\ &= \left(e^{\ln(n)}\right)^{\ln(\ln(n))} \\ &= n^{\ln(\ln(n))}. \end{aligned}$$

[1] In general, $f(x)^a = (f(x))^a$. In particular, $\ln(n)^{\ln(n)} = (\ln(n))^{\ln(n)}$.

Note that $\ln(\ln(n)) \geq 2$ if $n \geq e^{e^2}$. Thus, if $n \geq e^{e^2}$,

$$\frac{1}{\ln(n)^{\ln(n)}} = \frac{1}{n^{\ln(\ln(n))}} \leq \frac{1}{n^2}.$$

We know $\sum_{n=e^{e^2}} \frac{1}{n^2}$ converges since it is a p-series, hence so does $\sum_{n=e^{e^2}} \frac{1}{\ln(n)^{\ln(n)}}$ by the comparison test. Truncating a series does not change convergence properties, therefore the given series **converges**.