Quiz 4

2. (3 points) Determine whether the series $\sum_{n=1}^{\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2+1}\right)$ converges or diverges.

We first use the divergence test since it's unclear whether the terms of the series actually approach 0.

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2+1}\right) &= \lim_{n\to\infty}\left(\sqrt{n^2+2n}-\sqrt{n^2+1}\right)\frac{\sqrt{n^2+2n}+\sqrt{n^2+1}}{\sqrt{n^2+2n}+\sqrt{n^2+1}} \\ &= \lim_{n\to\infty}\frac{(n^2+2n)-(n^2+1)}{\sqrt{n^2+2n}+\sqrt{n^2+1}} \\ &= \lim_{n\to\infty}\frac{2n-1}{\sqrt{n^2+2n}+\sqrt{n^2+1}} \\ &= \lim_{n\to\infty}\frac{2n-1}{\sqrt{n^2+2n}+\sqrt{n^2+1}}\cdot\frac{1/n}{1/n} \\ &= \lim_{n\to\infty}\frac{2-\frac{1}{n}}{\sqrt{1+\frac{2}{n}}+\sqrt{1+\frac{1}{n}}} \\ &= \frac{2-0}{\sqrt{1+0}+\sqrt{1+0}} = 1.\end{aligned}$$

Since this is not equal to zero, by the divergence test the series **diverges**.

3. (3 points) Determine whether the series $\sum_{n=0}^{\infty}\frac{\sqrt{n}}{n^3+1}$ converges or diverges.

There are a number of ways to deal with this. Perhaps the swiftest is to notice, for $n \geq 1$, that

$$0 \leq \frac{\sqrt{n}}{n^3+1} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

We know $\sum_{n=1}\frac{1}{n^{5/2}}$ converges since it is a p-series with $p = 5/2$, and $5/2 > 1$. So, by the comparison test, $\sum_{n=1}^{\infty}\frac{\sqrt{n}}{n^3+1}$ converges, too. It doesn't matter what the starting value for n is when determining convergence, so the series in question converges.

Another method is to use the limit comparison test with $\frac{1}{n^{5/2}}$, which you might think of using because the numerator is $n^{1/2}$ and the denominator is basically n^3.

Yet another is the integral test. A complication is that $\frac{\sqrt{x}}{x^3+1}$ is an *increasing* function on $[0,1]$. However, this function is indeed monotonically decreasing on $[1,\infty)$, and all that really matters for the integral test is whether the function is *eventually* monotonically decreasing. So, we will compute $\int_1^{\infty}\frac{\sqrt{x}}{x^3+1}\,dx$. After having worked this out, it turns out the following substitution works well:[a] $u^2 = x^3$. The differential is $2u\,du = 3x^2\,dx$, and since $x^2 = u^{4/3}$, we have $dx = \frac{2}{3}u^{-1/3}\,du$. Hence,

$$\begin{aligned}\int_1^\infty \frac{\sqrt{x}}{x^3+1}\,dx &= \int_1^\infty \frac{u^{1/3}}{u^2+1}\cdot\tfrac{2}{3}u^{-1/3}\,du \\ &= \tfrac{2}{3}\int_1^\infty \frac{1}{u^2+1}\,du \\ &= \tfrac{2}{3}\lim_{b\to\infty}\left[\arctan u\right]_1^b \\ &= \tfrac{2}{3}\left(\tfrac{\pi}{2}-\tfrac{\pi}{4}\right) = \tfrac{\pi}{6}\end{aligned}$$

Since this improper integral converged, by the integral test, the original series **converges**.

[a] I'm saying that I solved this by other means, then realized afterwards that this would have been a good substitution. There is no way I would have thought of this ahead of time.

4. (3 points) Determine whether the series $\displaystyle\sum_{n=2}^{\infty}\frac{1+\cos(n)}{n^3\sqrt{n^2-1}}$ converges or diverges.

Again, we can use a comparison test. Note that if $n \geq 2$, then $\sqrt{n^2-1} \geq 1$, so

$$0 \leq \frac{1+\cos(n)}{n^3\sqrt{n^2-1}} \leq \frac{1+\cos(n)}{n^3}.$$

Secondly, $-1 \leq \cos(n) \leq 1$, so $0 \leq 1+\cos(n) \leq 2$, hence

$$\frac{1+\cos(n)}{n^3} \leq \frac{2}{n^3}.$$

Since $\sum_{n=2}^{\infty}\frac{2}{n^3}$ converges (it is a p-series with $p = 3 > 1$), then by the comparison test the given series **converges**.