Quiz 2

1. (5 points) Evaluate the definite integral $\displaystyle\int_2^3 \frac{x+3}{x^2-1}\,dx$.

The denominator factors as $x^2-1=(x-1)(x+1)$. The *ansatz* for partial fractions is

$$\frac{x+3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Multiplying through by $(x-1)(x+1)$, this is the equation $x+3 = A(x+1)+B(x-1)$. Since this is an equation that must be true for all values of x, it must be true for $x=-1$ and $x=1$. At $x=-1$, the equation is $2=-2B$, so $B=-1$. At $x=1$, the equation is $4=2A$, so $A=2$. Hence,

$$\begin{aligned}\int_2^3 \frac{x+3}{x^2-1}\,dx &= \int_2^3 \left(\frac{2}{x-1} - \frac{1}{x+1}\right) dx \\ &= \Big[2\ln|x-1| - \ln|x+1|\Big]_2^3 = \left[\ln\left|\frac{(x-1)^2}{x+1}\right|\right]_2^3 \\ &= \ln\frac{4}{4} - \ln\frac{1}{3} \\ &= \boxed{\ln 3}.\end{aligned}$$

2. (5 points) Evaluate the indefinite integral $\displaystyle\int x^3\sqrt{1-x^2}\,dx$.

Let $x=\sin(\theta)$, so $dx = \cos(\theta)\,d\theta$. With this substitution, $\sqrt{1-x^2}=\cos(\theta)$ since $1-\sin(\theta)^2 = \cos(\theta)^2$. Hence,

$$\int x^3\sqrt{1-x^2}\,dx = \int \sin(\theta)^3\sqrt{1-\sin(\theta)^2}\cos(\theta)\,d\theta. \qquad = \int \sin(\theta)^3\cos(\theta)^2\,d\theta.$$

This is a trigonometric integral, and according to the book the substitution $u=\cos(\theta)$ will work. Then $du = -\sin(\theta)\,d\theta$. Transforming then substituting, we have

$$\begin{aligned}\int \sin(\theta)^3\cos(\theta)^2\,d\theta &= \int (1-\cos(\theta)^2)\cos(\theta)^2\sin(\theta)\,d\theta \\ &= -\int (1-u^2)u^2\,du = \int (-u^2+u^4)\,du \\ &= -\tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + C \\ &= -\tfrac{1}{3}\cos(\theta)^3 + \tfrac{1}{5}\cos(\theta)^5 + C \\ &= -\tfrac{1}{3}\cos(\sin^{-1}(x))^3 + \tfrac{1}{5}\cos(\sin^{-1}(x))^5 + C.\end{aligned}$$

While this might be a fine answer, by considering triangles, $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, so we can put it into the equivalent form

$$\int x^3\sqrt{1-x^2}\,dx = \boxed{-\tfrac{1}{3}(1-x^2)^{3/2} + \tfrac{1}{5}(1-x^2)^{5/2} + C}.$$

Quiz 2

1. (5 points) Evaluate the integral $\int_{-1}^{1} \frac{4}{2-x^2}\,dx$.

The denominator factors as $2 - x^2 = (\sqrt{2}-x)(\sqrt{2}+x)$. The *ansatz* for partial fractions is

$$\frac{4}{(\sqrt{2}-x)(\sqrt{2}+x)} = \frac{A}{\sqrt{2}-x} + \frac{B}{\sqrt{2}+x}.$$

Multiplying through by $2-x^2$, this is the equation $4 = A(\sqrt{2}-x)+B(\sqrt{2}+x)$. Since this is an equation that must be true for all values of x, it must be true for $x=\sqrt{2}$ and $x=-\sqrt{2}$. At $x=\sqrt{2}$, the equation is $4 = 2\sqrt{2}B$, so $B=\sqrt{2}$. At $x=-\sqrt{2}$, the equation is $4=2\sqrt{2}A$, so $A=\sqrt{2}$. Hence,

$$\begin{aligned}\int_{-1}^{1}\frac{4}{2-x^2}\,dx &= \sqrt{2}\int_{-1}^{1}\left(\frac{1}{\sqrt{2}-x}+\frac{1}{\sqrt{2}+x}\right)dx \\ &= \sqrt{2}\Big[-\ln|\sqrt{2}-x| + \ln|\sqrt{2}+x|\Big]_{-1}^{1} \\ &= \sqrt{2}\Big((-\ln(\sqrt{2}-1)+\ln(\sqrt{2}+1)) - (-\ln(\sqrt{2}+1)+\ln(\sqrt{2}-1))\Big) \\ &= \boxed{2\sqrt{2}\Big(\ln(\sqrt{2}+1)-\ln(\sqrt{2}-1)\Big)}.\end{aligned}$$

2. (5 points) Evaluate the integral $\int \frac{\sqrt{x^2-1}}{x^3}\,dx$.

Let $x = \sec(\theta)$, so $dx = \sec(\theta)\tan(\theta)\,d\theta$. With this substitution, $\sqrt{x^2-1} = \tan(\theta)$ since $\sec(\theta)^2 - 1 = \tan(\theta)^2$. Hence,

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \int \frac{\sqrt{\sec(\theta)^2-1}}{\sec(\theta)^3}\sec(\theta)\tan(\theta)\,d\theta. = \int \frac{\tan(\theta)^2}{\sec(\theta)^2}\,d\theta.$$

In this case, it is worth expanding $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ and $\sec(\theta) = \frac{1}{\cos(\theta)}$, since the integral becomes

$$\begin{aligned}\int \frac{\tan(\theta)^2}{\sec(\theta)^2}\,d\theta = \int \sin(\theta)^2\,d\theta &= \int \tfrac{1}{2}(1-\cos(2\theta))\,d\theta \\ &= \tfrac{1}{2}\theta - \tfrac{1}{4}\sin(2\theta) + C.\end{aligned}$$

Substituting things back in, we have

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \tfrac{1}{2}\sec^{-1}(x) - \tfrac{1}{4}\sin(2\sec^{-1}(x)) + C.$$

While this answer is fine as-is, we can use some trig identities to eliminate the trigonometry in the second term. First, $\sin(2\sec^{-1}(x)) = 2\sin(\sec^{-1}(x))\cos(\sec^{-1}(x))$. Second, by considering triangles, $\sin(\sec^{-1}(x)) = x^{-1}\sqrt{x^2-1}$ and $\cos(\sec^{-1}(x)) = x^{-1}$. Hence,

$$\int \frac{\sqrt{x^2-1}}{x^3}\,dx = \boxed{\tfrac{1}{2}\sec^{-1}(x) - \tfrac{1}{2}x^{-2}\sqrt{x^2-1} + C}.$$