The irrationality of e

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In this project, we will show that Euler's constant e is irrational. For some background, recall the Taylor series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

where this series converges to e^x for every value of $x \in \mathbb{R}$. By substituting $x = 1$, we obtain a series representation of e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

We will show that e is irrational by way of contradiction. That is, by assuming we may write e as a fraction $\frac{a}{b}$ for integers a and b, we will will end up with an absurd conclusion.

Lemma 1. *If a and b are positive integers, then $b!\frac{a}{b}$ is an integer.*

Proof. Since $b! = b(b-1)!$, we see

$$b!\frac{a}{b} = b(b-1)!\frac{a}{b} = a(b-1)!,$$

and $a(b-1)!$ is an integer. □

Lemma 2. *Suppose n is a non-negative whole number such that $n \leq b$. Then $\frac{b!}{n!}$ is an integer.*

Proof. Since $n \leq b$, we can write $b! = b(b-1)(b-2)\cdots(n+1)n!$. Thus, $\frac{b!}{n!} = b(b-1)(b-2)\cdots(n+1)$, which is an integer. □

Lemma 3. *If $e = \frac{a}{b}$ for a and b positive integers, then $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is an integer.*

Proof. First, notice that

$$\begin{aligned}\sum_{n=b+1}^{\infty} \frac{b!}{n!} &= \left(\sum_{n=0}^{\infty} \frac{b!}{n!}\right) - \left(\sum_{n=0}^{b} \frac{b!}{n!}\right) \\ &= b!\left(\sum_{n=0}^{\infty} \frac{1}{n!}\right) - \left(\sum_{n=0}^{b} \frac{b!}{n!}\right) \\ &= b!e - \left(\sum_{n=0}^{b} \frac{b!}{n!}\right).\end{aligned}$$

For $b!e$, since we assumed $e = \frac{a}{b}$, then this is an integer by Lemma 1. And, for $\sum_{n=0}^{b} \frac{b!}{n!}$, since in each term we have $n \leq b$, this sum is an integer since each term is an integer by Lemma 2. The difference of integers is an integer, so therefore $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is an integer. □

Lemma 4. *For b a positive integer,*

$$\sum_{n=b+1}^{\infty} \frac{b!}{n!} < \sum_{n=1}^{\infty} \left(\frac{1}{b+1}\right)^n.$$

Proof. We have

$$\begin{aligned}
\sum_{n=b+1}^{\infty} \frac{b!}{n!} &= \frac{b!}{(b+1)!} + \frac{b!}{(b+2)!} + \frac{b!}{(b+3)!} + \cdots \\
&= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \\
&< \frac{1}{b+1} + \frac{1}{(b+1)(b+1)} + \frac{1}{(b+1)(b+1)(b+1)} + \cdots \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{b+1}\right)^n .
\end{aligned}$$

□

Lemma 5. *If $b \geq 2$ is an integer, then*

$$0 < \sum_{n=b+1}^{\infty} \frac{b!}{n!} < 1.$$

Proof. The inequality $0 < \sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is clear since each term of the series is positive. Applying Lemma 4, we have

$$\sum_{n=b+1}^{\infty} \frac{b!}{n!} < \sum_{n=1}^{\infty} \left(\frac{1}{b+1}\right)^n = \frac{\frac{1}{b+1}}{1 - \frac{1}{b+1}} = \frac{1}{b},$$

using the geometric series formula. Since $b \geq 2$ we have $\frac{1}{b} \leq \frac{1}{2} < 1$. This proves the lemma. □

Theorem 6. *The constant e is irrational.*

Proof. For sake of contradiction, suppose that $e = \frac{a}{b}$ for a and b positive constants. By Lemma 3, $\sum_{n=b+1}^{\infty} \frac{b!}{n!}$ is an integer. Also, since we know $2 < e < 3$, we may deduce $b \geq 2$, so we have that $0 < \sum_{n=b+1}^{\infty} \frac{b!}{n!} < 1$ by Lemma 5. But there are no integers that are strictly between 0 and 1, so this is a contradiction. Therefore, there are no positive integers a and b such that $e = \frac{a}{b}$, which is to say e is irrational. □