2. $y' + 2xy = 0 \qquad y = \sum_{n=0}^{\infty} c_n x^n$

$$0 = y' + 2xy = \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$(n-1 = m) \qquad (m = n+1)$

$$= \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m + \sum_{m=1}^{\infty} 2 c_{m-1} x^m$$

$$0 = c_1 + \sum_{m=1}^{\infty} \left((m+1) c_{m+1} + 2 c_{m-1} \right) x^m$$

$$\begin{cases} 0 = c_1 \\ 0 = (m+1) c_{m+1} + 2 c_{m-1} & m \geq 1 \end{cases}$$

recursion relation

$$c_{m+1} = \frac{-2 c_{m-1}}{m+1}$$

$(m+1 = n)$

$$(*) \quad c_n = \frac{-2 c_{n-2}}{n} \qquad n \geq 2$$

k	n	c_n
0	0	c_0
	1	$c_1 = 0$
1	2	$c_2 = \frac{-2c_0}{2} = \frac{-c_0}{1}$
	3	$c_3 = \frac{-2c_1}{3} = 0$
2	4	$c_4 = \frac{-2c_2}{4} = \frac{2^2 c_0}{4 \cdot 2} = \frac{c_0}{2 \cdot 1}$
	5	$c_5 = 0$
3	6	$c_6 = \frac{-2c_4}{6} = -\frac{2^3 c_0}{6 \cdot 4 \cdot 2} = \frac{-c_0}{3 \cdot 2 \cdot 1}$
	7	$c_7 = 0$
4	8	$c_8 = \frac{-2c_6}{8} = \frac{2^4 c_0}{8 \cdot 6 \cdot 4 \cdot 2} = \frac{c_0}{4 \cdot 3 \cdot 2 \cdot 1}$
	9	$c_9 = 0$

Guess: $c_{2k+1} = 0 \qquad k \geq 0$

$$c_{2k} = \frac{(-1)^k c_0}{k!}$$

Check $k=0$: $\frac{(-1)^0 c_0}{0!} = c_0$ ✓

Check k assuming $c_{2(k-1)} = \frac{(-1)^{k-1} c_0}{(k-1)!}$

$$c_{2k} \overset{(*)}{=} \frac{-2 c_{2k-2}}{2k} = \frac{-c_{2(k-1)}}{k} = \frac{-(-1)^{k-1} c_0}{k(k-1)!} = \frac{(-1)^k c_0}{k!} \; ✓$$

$$\boxed{y = \sum_{k=0}^{\infty} c_{2k} x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{k!} x^{2k}} = c_0 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$$

$$= c_0 e^{-x^2}$$

3. $(x+1)y' = 3y$

$0 = xy' + y' - 3y$

$$0 = \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 3 c_n x^n$$

$(m = n-1)$

$$0 = \sum_{n=0}^{\infty} n c_n x^n + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m - \sum_{n=0}^{\infty} 3 c_n x^n$$

$(m = n)$

$$0 = \sum_{n=0}^{\infty} \left(n c_n + (n+1) c_{n+1} - 3 c_n\right) x^n$$

$(n-3)c_n + (n+1)c_{n+1} = 0$ for $n \geq 0$

$$c_{n+1} = \frac{-(n-3)c_n}{n+1}$$

$(n+1 = m)$

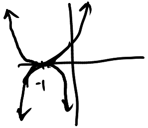
$$c_m = \frac{-(m-4)c_{m-1}}{m} \quad \text{for } m \geq 1$$

n	c_n
0	c_0
1	$c_1 = \frac{-(1-4)c_0}{1} = 3c_0$
2	$c_2 = \frac{-(2-4)c_1}{2} = c_1 = 3c_0$
3	$c_3 = \frac{-(3-4)c_2}{3} = \frac{c_2}{3} = c_0$
4	$c_4 = \frac{-(4-4)c_3}{4} = 0$
5	$c_5 = \frac{-(5-4)c_4}{5} = 0$
6	$c_6 = 0$
	$\vdots$

$$\boxed{y = c_0 + 3c_0 x + 3c_0 x^2 + c_0 x^3}$$

$y = c_0(1 + 3x + 3x^2 + x^3)$

$y = c_0(1+x)^3$



$(x+1)y' = 3y$

$$\int \frac{dy}{y} = \int \frac{3\,dx}{x+1}$$

$\ln|y| = 3\ln|x+1| + C$

$|y| = |x+1|^3 e^C$

$y = A|x+1|^3$

4. $y'' + xy' + y = 0$

$$0 = y'' + xy' + y$$

$$0 = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$(n-2 = m)$

$$0 = \sum_{m=0}^{\infty} (m+2)(m+1) c_{m+2} x^m + \sum_{n=0}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

$(m = n)$

$$0 = \sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} + n c_n + c_n \right) x^n$$

$$0 = (n+2)(n+1)c_{n+2} + (n+1)c_n \quad \text{for } n \geq 0$$

$$c_{n+2} = \frac{-\cancel{(n+1)}c_n}{(n+2)\cancel{(n+1)}} = \frac{-c_n}{n+2}$$

$(m = n+2)$

$$c_m = \frac{-c_{m-2}}{m} \quad \text{for } m \geq 2$$

k	n	c_n
	0	c_0
0	1	c_1
	2	$c_2 = \frac{-c_0}{2} = \frac{-c_0}{2^1 \cdot 1}$
1	3	$c_3 = \frac{-c_1}{3}$
	4	$c_4 = \frac{-c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{c_0}{2^2 \cdot 2 \cdot 1}$
2	5	$c_5 = \frac{-c_3}{5} = \frac{c_1}{5 \cdot 3}$
	6	$c_6 = \frac{-c_4}{6} = \frac{-c_0}{6 \cdot 4 \cdot 2} = \frac{-c_0}{2^3 \cdot 3 \cdot 2 \cdot 1}$
3	7	$c_7 = \frac{-c_5}{7} = \frac{-c_1}{7 \cdot 5 \cdot 3}$

Guess: $c_{2k} = \frac{(-1)^k c_0}{2^k k!}$

$$c_{2k+1} = \frac{(-1)^k c_1 2^k k!}{(2k+1)!}$$

$$y = \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1}$$

$$\boxed{y = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k!} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k 2^k k! x^{2k+1}}{(2k+1)!}}$$

$y = c_0 e^{-x^2/2} + c_1$???

$$7 \cdot 5 \cdot 3 \cdot 1 = \frac{6 \cdot 4 \cdot 2}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{2^3 \cdot 3 \cdot 2 \cdot 1}{7!} = \frac{2^3 3!}{7!}$$