1. $y' = x^2 y$

$$\int \frac{dy}{y} = \int x^2 dx$$

$$\ln|y| = \frac{1}{3}x^3 + C$$

$$y = A e^{\frac{1}{3}x^3}$$

n	c_n
0	c_0
1	0
2	0
3	$c_3 = \frac{c_0}{3}$
4	0
5	0
6	$c_6 = \frac{c_3}{6} = \frac{c_0}{6\cdot 3}$
7	0
8	0
9	$c_9 = \frac{c_6}{9} = \frac{c_0}{9\cdot 6\cdot 3}$

Now by series:

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$0 = y' - x^2 y = \sum_{n=1}^{\infty} n c_n x^{n-1} - x^2 \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2}$$

$n-1 = m$ $\qquad$ $m = n+2$

$$= \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m - \sum_{m=2}^{\infty} c_{m-2} x^m$$

$$= \underset{m=0}{c_1} + \underset{m=1}{2c_2 x} + \sum_{m=2}^{\infty} \left((m+1)c_{m+1} - c_{m-2}\right) x^m$$

$$\begin{cases} c_1 = 0 \\ 2c_2 = 0 \\ (m+1)c_{m+1} - c_{m-2} = 0 \quad \text{for } m \geq 2 \end{cases}$$

recursion relation

$$c_{m+1} = \frac{c_{m-2}}{m+1}$$

$(n = m+1)$

$$c_n = \frac{c_{n-3}}{n} \quad \text{for } n \geq 3$$

Guess: $c_{3k+1} = 0$ $\quad k \geq 0$

$c_{3k+2} = 0$

$$c_{3k} = \frac{c_0}{3\cdot 6\cdot 9\cdot \cdots \cdot (3k)}$$

$$= \frac{c_0}{(3\cdot 1)\cdot(3\cdot 2)\cdot(3\cdot 3)\cdots(3\cdot k)}$$

$$= \frac{c_0}{3^k \cdot 1\cdot 2\cdot 3\cdots k} = \frac{c_0}{3^k k!}$$

$$y = \sum_{k=0}^{\infty} \frac{c_0}{3^k k!} x^{3k} = \boxed{c_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^3}{3}\right)^k}$$

$$y = c_0 e^{x^3/3}$$

$$\int e^{-x^2} dx$$

$$y' = e^{-x^2}$$

2. $y' + 2xy = 0 \qquad \left(y = \sum_{n=0}^{\infty} c_n x^n\right)$

$$0 = \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$(m = n-1) \qquad (m = n+1)$

$$0 = \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m + \sum_{m=1}^{\infty} 2 c_{m-1} x^m$$

$$0 = c_1 + \sum_{m=1}^{\infty} \left((m+1)c_{m+1} + 2c_{m-1}\right) x^m$$

$$\begin{cases} 0 = c_1 \\ 0 = (m+1) c_{m+1} + 2 c_{m-1} \quad \text{for } m \geq 1 \end{cases}$$

$$c_{m+1} = -\frac{2c_{m-1}}{m+1}$$

$(m+1 = n)$

$$c_n = \frac{-2c_{n-2}}{n} \quad \text{for } n \geq 2$$

k	n	c_n
0	0	c_0
	1	0
1	2	$c_2 = \frac{-2}{2} \cdot c_0 = -c_0$
	3	0
2	4	$c_4 = \frac{-2}{4} \cdot c_2 = \frac{2^2}{4 \cdot 2} c_0 = \frac{c_0}{2 \cdot 1}$
	5	0
3	6	$c_6 = \frac{-2}{6} \cdot c_4 = \frac{-2^3}{6 \cdot 4 \cdot 2} c_0 = \frac{-c_0}{3 \cdot 2 \cdot 1}$
	7	0
4	8	$c_8 = \frac{-2}{8} \cdot c_6 = \frac{c_0}{4 \cdot 3 \cdot 2 \cdot 1}$

Guess: $c_{2k+1} = 0$ for $k \geq 0$

$$c_{2k} = \frac{(-1)^k c_0}{k!}$$

for $k=0$, $c_0 \stackrel{?}{=} \frac{(-1)^0 c_0}{0!} = c_0$ ✓

for $k \geq 1$ $\quad c_{2k} = \frac{-2c_{2k-2}}{2k} = \frac{-c_{2(k-1)}}{k}$

$$= \frac{-1}{k} \cdot \frac{(-1)^{k-1} c_0}{(k-1)!}$$

$$= \frac{(-1)^k c_0}{k!} \quad ✓$$

$$\boxed{y = \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{k!} x^{2k}} = c_0 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = c_0 e^{-x^2}$$

3. $(x+1)y' = 3y$

$$0 = xy' + y' - 3y$$

$$0 = x\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} 3 c_n x^n$$

$(m=n-1)$

$$0 = \sum_{n=1}^{\infty} n c_n x^n + \sum_{m=0}^{\infty} (m+1) c_{m+1} x^m - \sum_{n=0}^{\infty} 3 c_n x^n$$

$(m=n)$

$$0 = c_1 - 3c_0 + \sum_{n=1}^{\infty} \left(n c_n + (n+1)c_{n+1} - 3c_n\right) x^n$$

$$\begin{cases} 0 = c_1 - 3c_0 \\ 0 = (n-3)c_n + (n+1)c_{n+1} & \text{for } n \geq 1 \end{cases}$$

$$c_{n+1} = \frac{-(n-3)c_n}{n+1}$$

$m = n+1$

$$c_m = \frac{-(m-4)c_{m-1}}{m} \quad \text{for } m \geq 2$$

n	c_n
0	c_0
1	$c_1 = 3c_0$
2	$c_2 = \frac{-(-2)}{2}c_1 = \frac{-(-2)3c_0}{2} = 3c_0$
3	$c_3 = \frac{-(-1)}{3}c_2 = \frac{(-1)(-2)\cdot 3c_0}{3\cdot 2} = c_0$
4	$c_4 = \frac{-0\,c_3}{4} = 0$
5	$c_5 = \frac{-1\,c_4}{5} = 0$
6	$c_6 = 0$
	$\vdots$

$$y = c_0 + 3c_0 x + 3c_0 x^2 + c_0 x^3$$

$$y = c_0(1 + 3x + 3x^2 + x^3)$$

$$\boxed{y = c_0(1+x)^3}$$