Series solutions

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + c_{n+1} x^{n+1} + c_{n+2} x^{n+2} + \cdots$$

$$y' = 0 + c_1 + 2c_2 x + \cdots + n c_n x^{n-1} + (n+1) c_{n+1} x^n + (n+2) c_{n+2} x^{n+1} + \cdots = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = 0 + 0 + 2c_2 + \cdots + n(n-1) c_n x^{n-2} + (n+1) n c_{n+1} x^{n-1} + (n+2)(n+1) c_{n+2} x^n + \cdots = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

<u>ex</u> $y' = ay$, $a \in \mathbb{R}$

$$0 = y' - ay = \left(\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\right) - a\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \left((n+1) c_{n+1} - a c_n\right) x^n$$

so $(n+1) c_{n+1} - a c_n = 0$ for $n \geq 0$

$$c_{n+1} = \frac{a}{n+1} c_n$$

recursion relation

recurrence

n	c_n
0	c_0
1	$c_1 = \frac{a}{0+1} c_0$
2	$c_2 = \frac{a}{1+1} c_1 = \frac{a^2}{2 \cdot 1} c_0$
3	$c_3 = \frac{a}{2+1} c_2 = \frac{a^3}{3 \cdot 2 \cdot 1} c_0$

Guess: $c_n = \frac{a^n}{n!} c_0$

1. $c_0 \stackrel{?}{=} \frac{a^0}{0!} c_0 = c_0$ ✓

2. $n \geq 1$, $c_n \stackrel{?}{=} \frac{a^n}{n!} c_0 = \frac{a}{n} \cdot \frac{a^{n-1}}{(n-1)!} c_0 = \frac{a}{n} c_{n-1} = c_n$ ✓

0 1 2 3 4 ...

(induction)

so: $y = \sum_{n=0}^{\infty} \frac{a^n}{n!} c_0 x^n$

$y = c_0 \sum_{n=0}^{\infty} \frac{(ax)^n}{n!} = c_0 e^{ax}$

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

y(0) = c_0, so c_0 corr. to initial value problem

ex $y'' + y = 0$

$$0 = y'' + y = \left(\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \left((n+2)(n+1)c_{n+2} + c_n\right) x^n$$

so $(n+2)(n+1)c_{n+2} + c_n = 0$ for $n \geq 0$

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)} \quad \text{for } n \geq 0$$

$$c_n = \frac{-c_{n-2}}{n(n-1)} \quad \text{for } n \geq 2$$

n	c_n
0	c_0
1	c_1
2	$c_2 = \frac{-c_0}{2\cdot 1}$
3	$c_3 = \frac{-c_1}{3\cdot 2}$
4	$c_4 = \frac{-c_2}{4\cdot 3} = \frac{c_0}{4\cdot 3\cdot 2\cdot 1}$
5	$c_5 = \frac{-c_3}{5\cdot 4} = \frac{c_1}{5\cdot 4\cdot 3\cdot 2}$
6	$c_6 = \frac{-c_4}{6\cdot 5} = \frac{-c_0}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$
7	$c_7 = \frac{-c_5}{7\cdot 6} = \frac{-c_1}{7!}$

Guess: $c_{2k} = \frac{(-1)^k c_0}{(2k)!}$

$c_{2k+1} = \frac{(-1)^k c_1}{(2k+1)!}$

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k c_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k c_1}{(2k+1)!} x^{2k+1}$$

$$y = c_0 \cos(x) + c_1 \sin(x)$$

$$\left(y = y(0)\cos(x) + y'(0)\sin(x)\right)$$

$y(0) = c_0$

$y'(0) = c_1$

ex $y' = x^2 y$

$$y' - x^2 y = 0$$

$$P = -x^2 \qquad Q = 0 \qquad I = e^{\int P\,dx} = e^{\int -x^2\,dx} = e^{-\frac{1}{3}x^3}$$

$$y = \frac{1}{I}\int I Q\,dx = e^{\frac{1}{3}x^3}\int 0\,dx = Ce^{\frac{1}{3}x^3}$$

$$0 = y' - x^2 y = \left(\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n\right) - x^2\left(\sum_{n=0}^{\infty} c_n x^n\right) = \overset{n=0}{c_1} + \overset{n=1}{2c_2 x} + \sum_{n=2}^{\infty}\left((n+1)c_{n+1} - c_{n-2}\right)x^n$$

$$x^2\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} c_n x^{n+2} = c_0x^2 + c_1x^3 + c_2x^4 + \cdots = \sum_{n=2}^{\infty} c_{n-2}x^n$$

$$\begin{cases} c_1 = 0 \\ 2c_2 = 0 \\ (n+1)c_{n+1} - c_{n-2} = 0 & \text{for } n \geq 2 \end{cases}$$

$$c_{n+1} = \frac{c_{n-2}}{n+1} \quad \text{for } n \geq 2$$

$$c_n = \frac{c_{n-3}}{n} \quad \text{for } n \geq 3$$

n	c_n
0	c_0
1	$c_1 = 0$
2	$c_2 = 0$
3	$c_3 = \frac{c_0}{3}$
4	$c_4 = \frac{c_1}{4} = 0$
5	$c_5 = \frac{c_2}{5} = 0$
6	$c_6 = \frac{c_3}{6} = \frac{c_0}{6\cdot 3}$
7	0
8	0
9	$c_9 = \frac{c_6}{9} = \frac{c_0}{9\cdot 6\cdot 3}$

Guess $c_{3k} = \frac{c_0}{3\cdot 6\cdot 9\cdot \ldots \cdot (3k)}$

$$= \frac{c_0}{(3\cdot 1)\cdot(3\cdot 2)\cdot(3\cdot 3)\cdot \ldots \cdot(3\cdot k)} = \frac{c_0}{3^k\cdot 1\cdot 2\cdot 3\cdot \ldots \cdot k} = \frac{c_0}{3^k k!}$$

$$y = \sum_{k=0}^{\infty}\frac{c_0}{3^k k!}x^{3k} = c_0\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{x^3}{3}\right)^k = c_0 e^{\frac{1}{3}x^3}$$