17.4 - Series solutions

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_n x^n + c_{n+1} x^{n+1} + \cdots$$

$$y' = 0 + c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1} + (n+1) c_{n+1} x^n + (n+2) c_{n+2} x^{n+1} + \cdots$$

$$= \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$$

$$y'' = 0 + 0 + 2c_2 + 6c_3 + \cdots + n(n-1) c_n x^{n-2} + (n+1) n c_{n+1} x^{n-1} + (n+2)(n+1) c_{n+2} x^n + \cdots$$

$$= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$y' = \lambda y \quad (\lambda \in \mathbb{R})$$

$$\sum_{n=0}^{\infty} 0 x^n = 0 = y' - \lambda y = \left(\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n\right) - \lambda\left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \left((n+1)c_{n+1} - \lambda c_n\right) x^n$$

so $0 = (n+1) c_{n+1} - \lambda c_n$ for all n

$$c_{n+1} = \frac{\lambda c_n}{n+1}$$ recursion relation / recurrence

$$\boxed{\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^n \\ y' &= \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n \\ y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \end{aligned}}$$

$c_0\ ,\quad \underset{(n=0)}{c_1} = \frac{\lambda c_0}{0+1} = \lambda c_0\ ,\quad \underset{(n=1)}{c_2} = \frac{\lambda c_1}{1+1} = \frac{\lambda^2 c_0}{2}\ ,\quad \underset{n=2}{c_3} = \frac{\lambda c_2}{2+1} = \frac{\lambda^3 c_0}{3\cdot 2\cdot 1}\ ,\quad \underset{n=3}{c_4} = \frac{\lambda c_3}{3+1} = \frac{\lambda^4 c_0}{4\cdot 3\cdot 2\cdot 1}$

Guess: $c_n = \dfrac{\lambda^n c_0}{n!}$ for all n.

1) $n=0$: $\quad c_0 \overset{?}{=} \dfrac{\lambda^0 c_0}{0!} = \dfrac{1\cdot c_0}{1} = c_0$ ✓

2) $n \geq 1$: $\quad c_n \overset{?}{=} \dfrac{\lambda^n c_0}{n!} = \dfrac{\lambda}{n}\cdot\dfrac{\lambda^{n-1} c_0}{(n-1)!} = \dfrac{\lambda}{n}\cdot c_{n-1} = \dfrac{\lambda c_{n-1}}{(n-1)+1} = c_n$ ✓

So: soln to $y' = y$ is $y = \displaystyle\sum_{n=0}^{\infty} \frac{\lambda^n c_0}{n!} x^n = c_0 \sum_{n=0}^{\infty} \frac{\lambda^n x^n}{n!}$ $\qquad y(0) = \displaystyle\sum_{n=0}^{\infty} c_n 0^n = c_0$

where $c_0 = y_0$ (for inititial value problem)

$$y'' + y = 0$$

$$0 = y'' + y = \left(\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n\right) + \left(\sum_{n=0}^{\infty} c_n x^n\right) = \sum_{n=0}^{\infty} \left((n+2)(n+1) c_{n+2} + c_n\right) x^n$$

so $(n+2)(n+1) c_{n+2} + c_n = 0$ for all n

$$c_{n+2} = \frac{-c_n}{(n+2)(n+1)}$$

n	c_n
0	c_0
1	c_1
2	$c_2 = \frac{-c_0}{(0+2)(0+1)} = \frac{-c_0}{2\cdot 1}$
3	$c_3 = \frac{-c_1}{(1+2)(1+1)} = \frac{-c_1}{3\cdot 2}$
4	$c_4 = \frac{-c_2}{(2+2)(2+1)} = \frac{-c_2}{4\cdot 3} = \frac{c_0}{4\cdot 3\cdot 2\cdot 1}$
5	$c_5 = \frac{-c_3}{(3+2)(3+1)} = \frac{-c_3}{5\cdot 4} = \frac{c_1}{5\cdot 4\cdot 3\cdot 2}$

$$c_{2n} = \frac{(-1)^n c_0}{(2n)!} \qquad c_{2n+1} = \frac{(-1)^n c_1}{(2n+1)!}$$

$$y = \left(\sum_{n=0}^{\infty} \frac{(-1)^n c_0}{(2n)!} x^{2n}\right) + \left(\sum_{n=0}^{\infty} \frac{(-1)^n c_1}{(2n+1)!} x^{2n+1}\right)$$

$$y = c_0 \cos(x) + c_1 \sin(x)$$

($c_0 = y(0)$, $c_1 = y'(0)$ initial value problem)

$y' = x^2 y$

$$0 = y' - x^2 y = \left(\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n\right) - \underbrace{x^2\left(\sum_{n=0}^{\infty} c_n x^n\right)} = \left(\sum_{n=0}^{\infty}(n+1)c_{n+1}x^n\right) - \left(\sum_{n=2}^{\infty} c_{n-2}x^n\right)$$

$$\sum_{n=0}^{\infty} c_n x^{n+2} = \sum_{n=2}^{\infty} c_{n-2}x^n$$

$$c_0x^2 + c_1x^3 + c_2x^4 + \cdots \quad = \quad c_0x^2 + c_1x^3 + c_2x^4$$

$$= c_1 + 2c_2x + \sum_{n=2}^{\infty}\left((n+1)c_{n+1} - c_{n-2}\right)x^n$$

$c_1 = 0$

$2c_2 = 0$

$(n+1)c_{n+1} - c_{n-2} = 0$ for all $n \geq 2$

$$c_{n+1} = \frac{c_{n-2}}{n+1}$$

n	c_n
0	c_0
1	$c_1 = 0$
2	$c_2 = 0$
3	$c_3 = \frac{c_{2-2}}{2+1} = \frac{c_0}{3}$
4	$c_4 = \frac{c_{3-2}}{3+1} = 0$
5	$c_5 = \frac{c_{4-2}}{4+1} = 0$
6	$c_6 = \frac{c_{5-2}}{5+1} = \frac{c_0}{6\cdot 3}$
9	$c_9 = \frac{c_{8-2}}{8+1} = \frac{c_0}{9\cdot 6\cdot 3}$

Guess: $$c_{3k} = \frac{c_0}{(3k)\cdot(3k-3)\cdot(3k-6)\cdot\cdots\cdot 3} \quad \text{for } k \geq 1$$

$$= \frac{c_0}{3\cdot 6\cdot 9\cdot\cdots\cdot(3k)}$$

$$y = c_0\sum_{k=0}^{\infty}\frac{1}{3\cdot 6\cdot 9\cdot\cdots\cdot(3k)}x^{3k}$$

$$\boxed{\int \frac{dy}{y} = \int x^2\,dx}$$