(*) $ay'' + by' + cy = g(t)$

$\downarrow$ complementary eqn

(**) $ay'' + by' + cy = 0$

$\downarrow$ auxiliary polynomial

$a\lambda^2 + b\lambda + c = 0$

$\downarrow$ roots

$\lambda = r_1, r_2$

Case I) $r_1, r_2 \in \mathbb{R}$ and distinct

(**) has soln $y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

Case II) $r_1 = r_2 = r$

(**) has soln $y = (C_1 + C_2 t) e^{rt}$

Case III) $r_1, r_2 = \alpha \pm \beta i \quad (\beta \neq 0)$

(**) has soln $y = (c_1 \cos(\beta t) + C_2 \sin(\beta t)) e^{\alpha t}$

Then,

soln of (**)

$y = y_p + y_h$

Now, (A) $g(t) = (A_0 + A_1 t + \cdots + A_n t^n) e^{kt}$

Guess $y_p = (B_0 + B_1 t + \cdots + B_n t^n) e^{kt}$ if $k \neq r_1, r_2$

else, $y_p = (B_1 t + B_2 t^2 + \cdots + B_{n+1} t^{n+1}) e^{kt}$

(B) $g(t) = (A_0 + A_1 t + \cdots + A_n t^n) \cos(\gamma t) e^{\delta t}$

Guess $y_p = \big((B_0 + B_1 t + \cdots + B_n t^n) \cos(\gamma t)$
$+ (C_0 + C_1 t + \cdots + C_n t^n) \sin(\gamma t)\big) e^{\delta t}$ if $\gamma \pm \delta i \neq r_1, r_2$

(Otherwise mult by t)

ex $9y'' + y = e^{2t}$

$\downarrow$

$9y'' + y = 0$

$9\lambda^2 + 1 = 0$

$\lambda^2 = \frac{1}{9}$

$\lambda = \pm\frac{1}{3}i$

$y_h = C_1\cos(\tfrac{1}{3}t) + C_2\sin(\tfrac{1}{3}t)$

$y_p = Ae^{2x}$

$y_p' = 2Ae^{2x}$

$y_p'' = 4Ae^{2x}$

$9y_p'' + y_p = (9\cdot 4A + A)e^{2x}$

$= 37Ae^{2x} = e^{2x}$

$A = \frac{1}{37}$

$$\boxed{y = \frac{1}{37}e^{2x} + C_1\cos(\tfrac{1}{3}t) + C_2\sin(\tfrac{1}{3}t)}$$

ex $y'' - 4y' + 5y = e^{-x}$

$\lambda^2 - 4\lambda + 5 = 0$

$\lambda = \frac{4 \pm \sqrt{16 - 4\cdot 5}}{2} = 2 \pm i$

$y_h = e^{2x}(C_1\cos(x) + C_2\sin(x))$

$y_p = Ae^{-x}$

$y_p' = -Ae^{-x}$

$y_p'' = Ae^{-x}$

$(Ae^{-x}) - 4(-Ae^{-x}) + 5(Ae^{-x})$

$= 10Ae^{-x} = e^{-x}$

$A = \frac{1}{10}$

$$\boxed{y = \frac{1}{10}e^{-x} + e^{2x}(C_1\cos(x) + C_2\sin(x))}$$

ex (***) $y'' - 4y' + 4y = t - \sin(t)$

$\rightarrow$ (*) $y'' - 4y' + 4y = t$

$\rightarrow$ (**) $y'' - 4y' + 4y = -\sin(t)$

corr. to $r = \pm i$

$$\lambda^2 - 4\lambda + 4 = 0$$
$$(\lambda - 2)^2 = 0$$
$$\lambda = 2, 2$$
$$y_h = (c_1 + c_2 t)e^{2t}$$

(*) $y_p = (B_0 + B_1 t)e^{0t}$

$y_p' = B_1$

$y_p'' = 0$

$$(0) - 4(B_1) + 4(B_0 + B_1 t)$$
$$= 4(B_0 - B_1) + 4B_1 t = t$$

$$\begin{cases} 4B_1 = 1 \\ 4(B_0 - B_1) = 0 \end{cases} \qquad B_1 = \frac{1}{4}, \quad B_0 = \frac{1}{4}$$

$$y_p = \frac{1}{4} + \frac{1}{4}t$$

(**) $y_p = A\cos(t) + B\sin(t)$

$y_p' = -A\sin(t) + B\cos(t)$

$y_p'' = -A\cos(t) - B\sin(t)$

$$y_p'' - 4y_p' + 4y_p = \cos(t)(-A - 4B + 4A) + \sin(t)(-B + 4A + 4B) = -\sin(t)$$

$$\begin{cases} 3A - 4B = 0 \\ 4A + 3B = -1 \end{cases} \rightarrow \begin{cases} 7A - B = 0 \\ 4A + 3B = -1 \end{cases}$$

$$\rightarrow \begin{cases} 7A - B = 0 \\ 25A + 0 = -1 \end{cases} \qquad A = \frac{-1}{25}$$

$$B = 7A = -\frac{7}{25}$$

$$y_p = \frac{-1}{25}\cos(t) - \frac{7}{25}\sin(t)$$

(***) $\boxed{y = \frac{1}{4} + \frac{1}{4}t - \frac{1}{25}\cos(t) - \frac{7}{25}\sin(t) + (c_1 + c_2 t)e^{2t}}$

ex

$y'' - 4y = e^{2x}$

$\lambda^2 - 4 = 0$

$\lambda = \pm 2 \longrightarrow 2$ in list of roots

$y_p = Axe^{2x}$

$y_p' = A(e^{2x} + 2xe^{2x})$

$y_p'' = A(2e^{2x} + 2e^{2x} + 4xe^{2x})$

$= A(4e^{2x} + 4xe^{2x})$

$y_p'' - 4y_p = e^{2x}(4A) - \cancel{xe^{2x}(4A - 4A)} = e^{2x}$

$A = \frac{1}{4}$

$$\boxed{y = \frac{1}{4}xe^{2x} + C_1 e^{2x} + C_2 e^{-2x}}$$

ex

$y'' - 2y' + 5y = \sin(x)$

$\left.\begin{array}{l} y(0) = 1 \\ y'(0) = 1 \end{array}\right\} (*)$

$\lambda^2 - 2\lambda + 5 = 0$

$\lambda = \dfrac{2 \pm \sqrt{4 - 4 \cdot 5}}{2} = 1 \pm 2i$

$y_h = (C_1 \cos(2x) + C_2 \sin(2x))e^x$

$y_p = A\cos(x) + B\sin(x)$

$y_p' = -A\sin(x) + B\cos(x)$

$y_p'' = -A\cos(x) - B\sin(x)$

$y_p'' - 2y_p' + 5y_p = \cos(x)(-A - 2(B) + 5A)$
$+ \sin(x)(-B - 2(-A) + 5B)$

$\begin{cases} 4A - 2B = 0 \\ 2A + 4B = 1 \end{cases} \rightarrow \begin{cases} 2A - B = 0 \\ 2A + 4B = 1 \end{cases} \rightarrow \begin{cases} 2A - B = 0 \\ 0 + 5B = 1 \end{cases}$

$B = \frac{1}{5}$ $\quad A = \frac{1}{10}$

Now: solve for C_1, C_2 given $(*)$