$$e^{i(a+b)} = e^{ia}e^{ib} = (\cos(a) + i\sin(a))(\cos(b) + i\sin(b))$$

$$e^{i(a+b)} = \cos(a+b) + i\sin(a+b)$$

$$= \cos(a)\cos(b) + \cos(a)\, i\sin(b) + i\sin(a)\cos(b) + i^2\sin(a)\sin(b)$$

$$= (\cos(a)\cos(b) - \sin(a)\sin(b)) + i(\cos(a)\sin(b) + \sin(a)\cos(b))$$

Real parts: $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$

Imag. parts: $\sin(a+b) = \cos(a)\sin(b) + \sin(a)\cos(b)$

These are the angle sum identities!

(no geometry, only calculus!)

de Moivre

$$e^{2ai} = (e^{ai})^2 = (\cos(a) + i\sin(a))^2$$

$$e^{2ai} = \cos(2a) + i\sin(2a)$$

$$= \cos(a)^2 + 2\cos(a)\, i\sin(a) + i^2\sin(a)^2$$

Real: $\cos(2a) = \cos(a)^2 - \sin(a)^2$

Imag: $\sin(2a) = 2\cos(a)\sin(a)$

Linear differential equations

first-order: $py' + qy = g$, p, q, g are fns of t

second-order: $py'' + qy' + ry = g$, p, q, r, g are fns of t

homogeneous: the $g=0$ case ("homog. linear diff. eq.")

with constant coefficients: p, q, r are constants

(*) inhomogeneous: $py'' + qy' + ry = g$

(**) homogeneous: $py'' + qy' + ry = 0$

Set of all solutions to (*) parameterized by c_1, c_2:

$$y = y_p + c_1 y_{h_1} + c_2 y_{h_2}$$

Lemma If y_p is a soln to (*) and y_h is a soln to (**) then $y = y_p + c y_h$ is a soln to (*).
c is a constant

Pf

$$\begin{aligned}
& py'' + qy' + ry \\
&= p(y_p + cy_h)'' + q(y_p + cy_h)' + r(y_p + cy_h) \\
&= p(y_p'' + cy_h'') + q(y_p' + cy_h') + r(y_p + cy_h) \\
&= (py_p'' + qy_p' + ry_p) + c(y_h'' + qy_h' + ry_h) \\
&= g + c \cdot 0 \\
&= g
\end{aligned}$$

, so y is a soln to (*) ∎

Lemma If y_1, y_2 are solns to (*), then $y = y_1 - y_2$ is a soln to (**).

So: every soln to (*) is $y_p + cy_h$ for some y_h soln to (**)

H.S.O.L.D.E.C.C.

$$ay'' + by' + cy = 0$$

Guess: $y = e^{\lambda t}$

$$\begin{aligned} 0 &= a(e^{\lambda t})'' + b(e^{\lambda t})' + c(e^{\lambda t}) \\ &= a\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + c e^{\lambda t} \\ &= (a\lambda^2 + b\lambda + c)e^{\lambda t} \end{aligned}$$

$e^{\lambda t} \neq 0$ for every t

$0 = a\lambda^2 + b\lambda + c$ ← auxiliary poly, characteristic poly

ex $y'' + y = 0$

$$\lambda^2 + 1 = 0$$
$$\lambda^2 = -1$$
$$\lambda = \pm i$$

$$(e^{it})'' + (e^{it}) = i^2 e^{it} + e^{it} = -e^{it} + e^{it} = 0$$

$$\begin{aligned} y &= c_1 e^{it} + c_2 e^{-it} \\ &= c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t)) \\ &= (c_1 + c_2)\cos(t) + (c_1 - c_2)i\sin(t) \end{aligned}$$

$\left(\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}\right)$

$\left(\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}\right)$

$A = c_1 + c_2$, $B = (c_1 - c_2)i$ $\qquad c_1 = \dfrac{A + Bi}{2}$ $\quad c_2 = \dfrac{A - Bi}{2}$

$$= A\cos(t) + B\sin(t)$$

$$ay' + by = 0$$
$$y' = \frac{-b}{a}y$$
$$y = Ae^{-bt/a}$$

ex $y'' - 3y' + 2y = 0$

$$\lambda^2 - 3\lambda + 2 = 0$$
$$(\lambda - 2)(\lambda - 1) = 0$$
$$\lambda = 1, 2$$

→ $y = e^t$ and $y = e^{2t}$ are solns.

$y = 0$ is particular soln

$y = 0 + \underbrace{c_1 e^t + c_2 e^{2t}}_{\text{linear combination}}$ for c_1, c_2 constants

$$y'' - 2y' + 1 = 0$$
$$\lambda^2 - 2\lambda + 1 = 0$$
$$(\lambda - 1)^2 = 0$$
$$\lambda = 1, 1$$
$$y = (c_0 + c_1 t)e^t$$