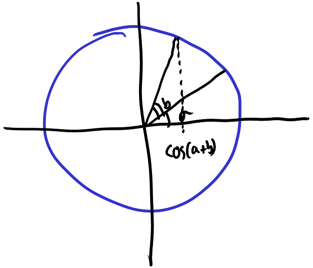
$$e^{i\theta} = \cos(\theta) + i\sin\theta$$



$$\begin{aligned}
\cos(a+b) &= \mathrm{Re}(\cos(a+b) + i\sin(a+b)) \\
&= \mathrm{Re}\left(e^{i(a+b)}\right) \\
&= \mathrm{Re}\left(e^{ia}e^{ib}\right) \\
&= \mathrm{Re}\left[(\cos(a) + i\sin(a))(\cos(b) + i\sin(b))\right] \\
&= \mathrm{Re}\left[\cos(a)\cos(b) + \cos(a)\, i\sin(b) \right. \\
&\qquad \left. + i\sin(a)\cos(b) + i^2\sin(a)\sin(b)\right]
\end{aligned}$$

$$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\sin(a+b) = \mathrm{Im}\left(e^{i(a+b)}\right) = \cos(a)\sin(b) + \sin(a)\cos(b)$$

Linear differential equations

first-order: $p y' + q y = g$ with p, q, g functions of t

$$\left(p(t)\, y'(t) + q(t)\, y(t) = g(t) \right)$$

second-order: $p y'' + q y' + r y = g$ with p, q, r, g fns of t

homogeneous: $g = 0$

Lemma If y_h is a soln to $py'' + qy' + ry = 0$ (**)
and y_p is a soln to $py'' + qy' + ry = g$ (*)
then $y = cy_h + y_p$ is a soln to (*), c a constant.

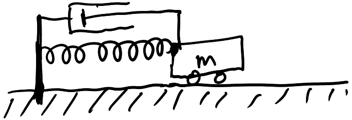
Pf
$$\begin{aligned} py'' + qy' + ry &= p(cy_h + y_p)'' + q(cy_h + y_p)' + r(cy_h + y_p) \\ &= p(cy_h'' + y_p'') + q(cy_h' + y_p') + r(cy_h + y_p) \\ &= c(py_h'' + qy_h' + ry_h) + (py_p'' + qy_p' + ry_p) \\ &= c \cdot 0 + g \\ &= g. \end{aligned}$$

So $y = cy_h + y_p$ is a soln to (*) $\square$

Lemma If y_1 and y_2 are solns to (*), then $y = y_1 - y_2$ is a soln to (**).

Takeaway: solving homog. eqns is very important for solving general ones.

Lemma If y_1 and y_2 are solns to (**), then $y = \underbrace{c_1 y_1 + c_2 y_2}_{\text{linear combination}}$ is as well



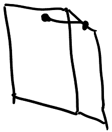
$x(t)$ = position of cart

$$m x'' = F = F_{spring} + F_{damper} = -kx - \alpha x'$$

$$F_{spring} = -kx$$

$$F_{damper} = -\alpha x'$$

$$\boxed{m x'' + \alpha x' + k x = 0}$$



Guess: $x = e^{ct}$

$$m(e^{ct})'' + \alpha(e^{ct})' + k(e^{ct})$$

$$= m c^2 e^{ct} + \alpha c e^{ct} + k e^{ct}$$

$$= (mc^2 + \alpha c + k) e^{ct}$$

$mc^2 + \alpha c + k = 0$ or $e^{ct} = 0$

never $= 0$

auxiliary equation
or characteristic eqn

$$c = \frac{-\alpha \pm \sqrt{\alpha^2 - 4mk}}{2m}$$

$$= \lambda_1, \lambda_2$$

$$x = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

<u>ex</u> $\alpha = 0$, $m = 1$.

$$x'' + kx = 0$$

$$\lambda^2 + k = 0$$

$$\lambda = \pm\sqrt{-k} = \pm i\sqrt{k}$$

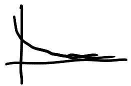
$$x = A_1 e^{it\sqrt{k}} + A_2 e^{-it\sqrt{k}} = \cdots = B_1 \cos(\sqrt{k}\, t) + B_2 \sin(\sqrt{k}\, t)$$

ex $\alpha=1,\ m=1$

$$x'' + x' + kx = 0$$
$$\lambda^2 + \lambda + k = 0$$
$$\lambda = \frac{-1 \pm \sqrt{1-4k}}{2}$$

Case I: $1-4k > 0 \rightsquigarrow \frac{1}{4} > k$

$$x = A_1 e^{\left(-\frac{1}{2} + \frac{\sqrt{1-4k}}{2}\right)t} + A_2 e^{\left(-\frac{1}{2} - \frac{\sqrt{1-4k}}{2}\right)t}$$



$$\frac{\sqrt{1-4k} - 1}{2}$$

$0 \leq k < \frac{1}{4}$

$0 \leq 4k < 1$

$0 \geq -4k > -1$

$1 \geq 1-4k > 0$