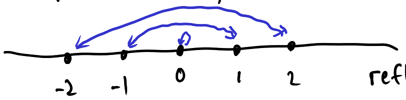
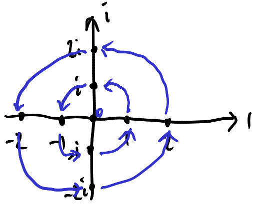
Complex numbers $i^2 = -1$

multiplication by -1



reflection through 0

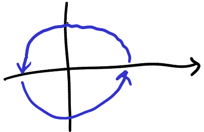
multiplication by i



rotation by 90° counterclockwise around 0

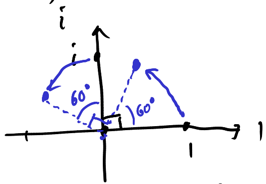
$(-1) = i^2 = i \cdot i$

$i^2 x = i(ix)$
rotate 90° CCW twice
i.e. rotation by 180°



multiplication by $e^{i\theta} = \cos\theta + i\sin\theta$

ex $\theta = \frac{\pi}{3}$



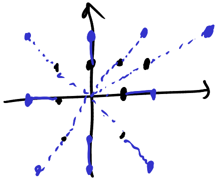
$e^{i\theta} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$e^{i\theta} i = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

rotation by θ CCW

multiplication by $r \in \mathbb{R}$, $r > 0$

ex $r = 2$



dilation/scaling by r

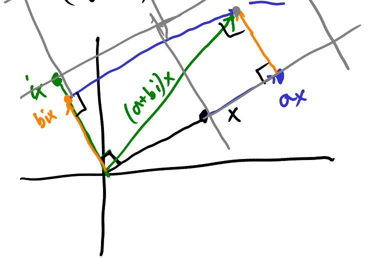
multiplication by $re^{i\theta}$, $r > 0$

scales by r and rotates by θ

$re^{i\theta} = e^{i\theta} r$

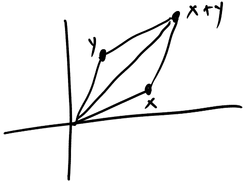
multiplication by a+bi

$(a+bi)\,x = ax + bix$



$a + b = c$

x + y



$a+bi$

$\updownarrow$

$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ $a,b\in\mathbb{R}$

Aside: A point of view of $\mathbb{C}$:

complex numbers are 2x2 matrices

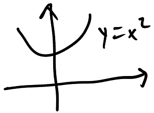
$1 \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad i \longleftrightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$e^{i\theta} \longleftrightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad re^{i\theta} \leftrightarrow \begin{pmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{pmatrix}$

$re^{i\theta} z + b$

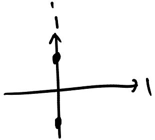
Polynomial roots

ex $x^2+5=0$



$x^2=-5$

$x=\pm\sqrt{-5}=\pm\sqrt{-1}\sqrt{5}=\pm i\sqrt{5}$



Fundamental theorem of algebra

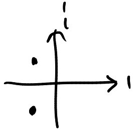
if $p(x)$ is a polynomial of degree ≥ 1 then it has a (complex) root.

$\Rightarrow \quad p(x)=a(x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)$

ex $x^2+x+1=0$

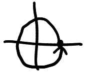
discriminant $= b^2-4ac = 1^2-4\cdot1\cdot1=-3<0$

$x=\dfrac{-1\pm\sqrt{1^2-4\cdot1\cdot1}}{2\cdot1}=\dfrac{-1\pm\sqrt{-3}}{2}=\dfrac{-1\pm\sqrt{3}\,i}{2}$



Conjugates: $a+bi$, $\overline{a+bi}=a-bi$

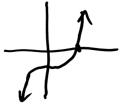
$e^{2\pi i}=1$



$e^{i\theta}\cdot e^{2\pi i}=e^{i\theta}$

$=e^{i(\theta+2\pi)}=e^{i(\theta+2\pi n)}$, $n\in\mathbb{Z}$

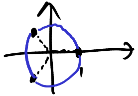
ex $x^3-1=0$



or: $x^3=1$ "x is cube root of unity" (1)

$(x-1)(x^2+x+1)=0$

$x=1,\ \dfrac{-1\pm\sqrt{3}\,i}{2}$



$x=re^{i\theta}$

$1\cdot e^{i0}=1=(re^{i\theta})^3=r^3e^{3i\theta}$

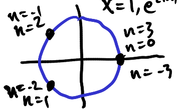
$e^{i0}=e^{3i\theta}$ $\xrightarrow{\ln}$ $i0=3i\theta+2\pi i n$, $n\in\mathbb{Z}$

$-2\pi n=3\theta$, $\theta=\dfrac{-2\pi n}{3}$

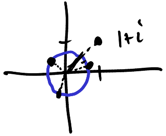
magnitudes: $1=r^3$, $r=1$ $(r\in\mathbb{R})$

$x=e^{-2\pi i n/3}$, $n\in\mathbb{Z}$

$x=1, e^{2\pi i/3}, e^{-2\pi i/3}$



Cube roots of $1+i$



$$1+i = \sqrt{1^2+1^2}\, e^{i\pi/4}$$
$$= \sqrt{2}\, e^{\pi i/4}$$

$$x^3 = \sqrt{2}\, e^{\pi i/4}$$

$$x = re^{i\theta}$$

$$r^3 e^{3i\theta} = \sqrt{2}\, e^{\pi i/4}$$

$$\begin{cases} r^3 = \sqrt{2} & \rightsquigarrow \quad r = (2^{1/2})^{1/3} = 2^{1/6} \\ e^{3i\theta} = e^{\pi i/4} & \rightsquigarrow \quad 3i\theta = \pi i/4 + 2\pi i n \quad n \in \mathbb{Z} \end{cases}$$

$$\theta = \frac{\pi}{12} + \frac{3\pi n}{2}$$

$$x = 2^{1/6} e^{i\left(\frac{\pi}{12} + \frac{3\pi n}{2}\right)}, \quad n \in \mathbb{Z}$$

$$x = 2^{1/6} e^{\pi i/12}, \quad 2^{1/6} e^{i(\pi/12 + 3\pi/2)}, \quad 2^{1/6} e^{i(\pi/12 + 3\pi)}$$

$$y = e^{it} \qquad y' = ie^{it} = iy$$

solves $y' = iy$ $\rightsquigarrow$ solns $y = Ae^{it}$