Appendix H – Complex numbers

not hard/difficult, rather two numbers put together

$$\underset{\substack{\text{naturals}\\0,1,2,3,\ldots}}{\mathbb{N}} \overset{\text{subtraction}}{\subset} \underset{\substack{\text{integers}\\ \ldots,-2,-1,0,1,2,\ldots}}{\mathbb{Z}} \overset{\text{division}}{\subset} \underset{\substack{\text{rationals}\\ \text{ex } 7/2}}{\mathbb{Q}} \overset{\text{limits}}{\subset} \underset{\substack{\text{reals}\\ \text{ex } \sqrt{2}}}{\mathbb{R}} \overset{\substack{\text{polynomial}\\ \text{roots}}}{\subset} \underset{\substack{\text{complex numbers}\\ \text{ex } x^2+1 \text{ roots}\\ i,-i}}{\mathbb{C}} \left(\subset \underset{\substack{\text{quaternions}\\ \text{(Hamilton)}\\ i,j,k}}{\mathbb{H}} \subset \underset{\text{octonions}}{\mathbb{O}} \right)$$

def $\mathbb{C}$ is the set of all things of the form $a+bi$, for $a, b \in \mathbb{R}$

"rectangular form"

$i := \sqrt{-1}$

How do we deal with i? $i^2 = -1$.

ex $(2+i)\,i = 2 \cdot i + i \cdot i = 2i - 1$

ex
$$\begin{aligned}(1+i)(1-i) &= 1 \cdot 1 + 1(-i) + i \cdot 1 + i(-i)\\ &= 1 - i + i - i^2\\ &= 1 - i + i - (-1) = 2\end{aligned}$$

ex $(\underbrace{1}_{\substack{\text{real}\\ \text{part}}} + \underbrace{2}_{\substack{\text{imaginary}\\ \text{part}}}i) + (3 - 4i) = 4 - 2i$

Electrical engineering: $a + \underbrace{b}_{\substack{j\text{imaginary}\\ \text{part}}}j$

Complex conjugate: $\overline{a+bi} = a - bi$

fact $|zw| = |z||w|$

subfact $\overline{zw} = \bar{z}\bar{w}$

$a+bi = a + b\sqrt{-1}$

$$(a+bi)\overline{(a+bi)} = (a+bi)(a-bi)$$
$$= a\cdot a + bi\cdot a + a(-bi) + bi(-bi)$$
$$= a^2 + abi - abi - b^2 i^2$$
$$= a^2 + b^2$$

ex $$\frac{2+i}{1-i}\cdot\frac{1+i}{1+i} = \frac{(2+i)(1+i)}{1^2+1^2} = \frac{2\cdot 1 + 2i + i\cdot 1 + i^2}{2} = \frac{2+3i-1}{2}$$
$$= \frac{1+3i}{2} = \frac{1}{2} + \frac{3}{2}i$$

absolute value / norm / modulus / magnitude / length

$|z| := \sqrt{z\bar{z}}$ $\qquad |a+bi| = \sqrt{a^2+b^2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$(ix)^n = i^n x^n$$

n	i^n
0	1
1	i
2	-1
3	$-i$
4	1
5	i
6	-1
7	$-i$
8	1

$$i^{n+1} = i \cdot i^n$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \frac{1}{0!} + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$$

$$= \frac{1}{0!} + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

$$= \left(\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

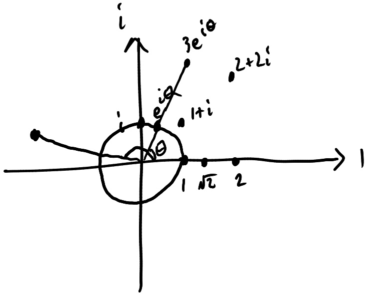
$$+ i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$e^{ix} = \cos(x) + i\sin(x)$$ ← Euler's identity

$$\left(e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1\right)$$

$$e^{a+bi} = e^a e^{bi} = e^a\cos(b) + e^a\sin(b)\,i$$

Argand diagram



$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$|z|$ = distance between 0 and z in plane

$$|a+bi| = \sqrt{a^2+b^2}$$

$$|e^{i\theta}| = |\cos\theta + i\sin\theta|$$
$$= \sqrt{\cos^2\theta + \sin^2\theta}$$
$$= \sqrt{1} = 1$$

$$|re^{i\theta}| = |r||e^{i\theta}| = r$$

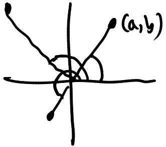
$$a+bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$$
$$= r\cos\theta + r\sin(\theta)\, i$$

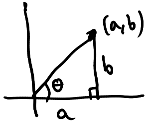
$$\begin{cases} a = r\cos\theta \\ b = r\sin\theta \end{cases}$$

1) $r = \sqrt{a^2+b^2}$

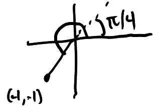
2) $\theta = \tan^{-1}(b,a)$

atan2, intraduced in 1970s in FORTRAN





$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$
$$= \tan^{-1}(b,a)$$



$$\tan^{-1}\left(\frac{-1}{-1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$
$$\tan^{-1}(-1,-1) = 5\pi/4$$

Problem 1. Write the number in polar form with argument between 0 and 2π.

1) $-2+2i$

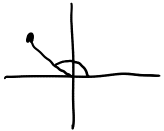
2) $-\sqrt{3}+i$

3) $3+3\sqrt{3}i$

$$(-2+2i)(-\sqrt{3}+i) = (2\sqrt{2}e^{3\pi i/4})(2e^{5\pi i/6})$$
$$= 4\sqrt{2}\, e^{3\pi i/4 + 5\pi i/6}$$
$$= 4\sqrt{2}\, e^{19\pi i/12}$$

$$(-2+2i)+(-\sqrt{3}+i) = (-2-\sqrt{3}) + 3i$$

1)

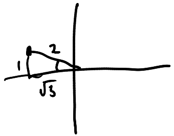


$$r = \sqrt{(-2)^2+2^2} = 2\sqrt{2}$$

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$-2+2i = 2\sqrt{2}\, e^{\frac{3\pi}{4}i} = 2\sqrt{2}\, e^{3\pi i/4}$$
$$= 2\sqrt{2}\exp(3\pi i/4)$$
$$(= 2\sqrt{2}\operatorname{cis}(3\pi/4))$$

2) $-\sqrt{3}+i$



$$r = \sqrt{(-\sqrt{3})^2+1^2} = 2$$

$$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$-\sqrt{3}+i = 2e^{5\pi i/6}$$