Appendix H – Complex numbers

not hard/difficult $\rightsquigarrow$ two numbers put together

$$\underset{\text{natural numbers}}{\mathbb{N}} \overset{\text{subtraction}}{\subset} \underset{\text{integers}}{\mathbb{Z}} \overset{\text{division}}{\subset} \underset{\text{rationals}}{\mathbb{Q}} \overset{\text{limits}}{\subset} \underset{\text{reals}}{\mathbb{R}} \overset{\text{polynomial roots}}{\subset} \underset{\text{complex numbers}}{\mathbb{C}} \quad (\subset \underset{\text{quaterions}}{\mathbb{H}})$$

natural numbers: $0, 1, 2, \dots$

integers: $\dots, -2, -1, 0, 1, 2, \dots$

rationals: ex $5/2$

reals: ex $\sqrt{2}$

complex numbers: ex roots of x^2+1

quaterions: i, j, k

x^2+1 has roots $i = \sqrt{-1}$ and $-i = -\sqrt{-1}$

(in electrical eng., $j = \sqrt{-1}$)

$$x^2+1=0$$
$$x^2=-1$$
$$x=\pm\sqrt{-1}$$

(rectangular form)

A complex number is $a+bi$ for $a, b \in \mathbb{R}$, where $i^2=-1$.

ex
$$i = 0+1i \qquad a = a+0i$$
$$1 = 1+0i \qquad (a \in \mathbb{R})$$

ex
$$(2+3i)\,i = 2\cdot i + 3i\cdot i$$
$$= 2i + 3\cdot(-1)$$
$$= -3+2i$$

ex
$$(1+i)(1-i) = 1\cdot 1 + i\cdot 1 + 1(-i) + \underbrace{i(-i)}_{(-1)i^2}$$
$$= 1 + i - i + 1$$
$$= 2+0i$$

(complex) Conjugate $\quad \overline{a+bi} = a-bi$

$$\overline{a+b\sqrt{-1}} = a-b\sqrt{-1}$$

ex $\quad \dfrac{1+i}{1-i}\cdot\dfrac{1+i}{1+i} = \dfrac{1^2+2\cdot 1\cdot i+i^2}{1^2+1^2} = \dfrac{1+2i-1}{2} = i$

$$(a+bi)\overline{(a+bi)} = (a+bi)(a-bi)$$
$$= a\cdot a + bi\cdot a + a(-bi) + \overbrace{bi(-bi)}^{(-1)b^2i^2}$$
$$= a^2 + abi - abi + b^2$$
$$= a^2+b^2$$

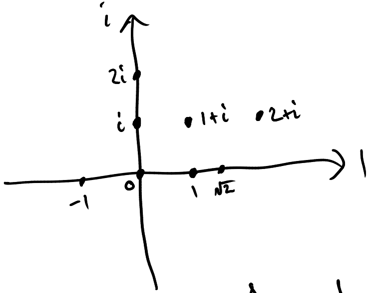
$$\underbrace{a}_{\text{real part}} + \underbrace{bi}_{\text{imaginary part}}$$

$$\left(a + \underbrace{bj}_{j\text{imaginary part}}\right)$$

norm / modulus / (length) / absolute value

$$|z| = \sqrt{z\bar{z}}$$

$$|a+bi| = \sqrt{a^2+b^2}$$



Argand plane

$$|zw| = |z||w|$$

Euler's formula

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \frac{1}{0!} + \frac{ix}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

$$= \left(\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$$

$$+ i\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\boxed{e^{ix} = \cos(x) + i\sin(x)}$$

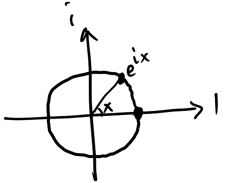
$$e^{a+bi} = e^a e^{bi} = e^a(\cos(b) + i\sin(b))$$

$$e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1$$

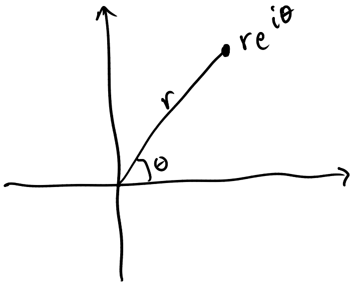
$$(ix)^n = i^n x^n$$

n	i^n
0	1
1	i
2	-1
3	$-i$
4	1
5	i
6	-1
7	$-i$
8	1

$$i^{n+1} = i \cdot i^n$$



polar form: $re^{i\theta} = r\cos(\theta) + i\,r\sin(\theta)$



$$|e^{i\theta}| = |\cos(\theta) + i\sin\theta|$$
$$= \sqrt{\cos^2(\theta) + \sin^2(\theta)}$$
$$= \sqrt{1} = 1$$

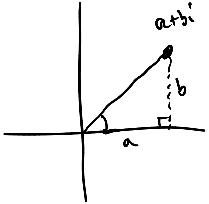
$$|re^{i\theta}| = |r|$$

$a+bi \rightsquigarrow$ polar form ?

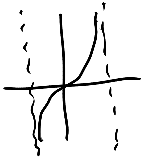
$$r = |a+bi|$$

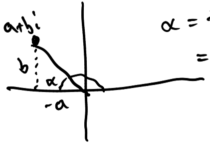
$\theta = \tan^{-1}\frac{b}{a}$ or something similar

$$\theta = \operatorname{atan2}(b, a)$$



$$\theta = \tan^{-1}\frac{b}{a}$$





$$\alpha = \tan^{-1}\frac{b}{-a}$$
$$= -\tan^{-1}\frac{b}{a}$$

$$\theta = \pi - \alpha = \pi + \tan^{-1}\frac{b}{a}$$

Problem 1. Write the number in polar form with argument between 0 and 2π.

1) $-2+2i$

2) $-\sqrt{3}+i$

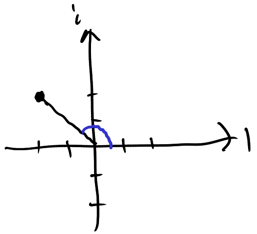
3) $3+3\sqrt{3}i$

$re^{i\theta}$

modulus

argument (angle)

1)



$$|-2+2i| = \sqrt{(-2)^2+2^2} = 2\sqrt{2}$$

$$\theta = \frac{3\pi}{4}$$

$$-2+2i = 2\sqrt{2}\, e^{i\cdot\frac{3\pi}{4}}$$

$$= 2\sqrt{2}\, e^{3\pi i/4}$$

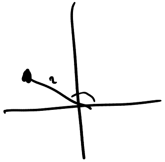
$$= 2\sqrt{2}\exp(3\pi i/4)$$

$$\left(= 2\sqrt{2}\,\mathrm{cis}(3\pi/4)\right)$$

2) $-\sqrt{3}+i$

$$|-\sqrt{3}+i| = \sqrt{\sqrt{3}^2+1^2}$$

$$= 2$$



$$\theta = \pi + \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right)$$

$$= \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= 5\pi/6$$

$$-\sqrt{3}+i = 2e^{5\pi i/6}$$

Polar, mult is easy: $(-2+2i)(-\sqrt{3}+i) = (2\sqrt{2}\,e^{3\pi i/4})(2e^{5\pi i/6}) = 4\sqrt{2}\,e^{3\pi i/4+5\pi i/6} = 4\sqrt{2}\,e^{19\pi i/12}$

rectangular, adding is easy: $(-2+2i)+(-\sqrt{3}+i) = (-2-\sqrt{3})+3i$