9.5 – Linear equations

Separable equations: $y' = f(x)g(y)$

$$y' + P(x)y = Q(x)$$

integrating factor $I(x) = \exp\left(\int P(x)\,dx\right)$

$$\frac{d}{dx}I = \frac{d}{dx}e^{\int P\,dx} = e^{\int P\,dx}\frac{d}{dx}\int P\,dx = e^{\int P\,dx}P = IP$$

Then $y = \frac{1}{I}\left(\int IQ\,dx + C\right)$

diff. eqn $\iff I\cdot(y' + Py) = I\cdot Q$

$\iff Iy' + IPy = IQ$

$\iff (Iy)' = IQ$

$\iff Iy = \int IQ\,dx + C$

$\iff y = \frac{1}{I}\left(\int IQ\,dx + C\right)$

ex $y' + y = 1$

$P(x) = 1 \quad Q(x) = 1$

$I = \exp\left(\int 1\,dx\right) = e^x$

$y = \frac{1}{e^x}\left(\int e^x \cdot 1\,dx + C\right)$

$= \frac{1}{e^x}\left(e^x + C\right) = 1 + Ce^{-x}$

$\frac{dy}{dx}$ $y' = 1 - y$

$\int \frac{dy}{1-y} = \int dx$

1. Solve the differential equation:

   a. $4x^3y + x^4y' = \sin^3 x$

$$y' + \frac{4}{x}y = \frac{\sin^3 x}{x^4}$$

$$P(x) = \frac{4}{x} \quad Q(x) = \frac{\sin^3 x}{x^4}$$

$$I = \exp\left(\int \frac{4}{x}\,dx\right) = \exp(4\ln|x|) = e^{4\ln|x|} = \left(e^{\ln|x|}\right)^4 = |x|^4 = x^4$$

$$Y = \frac{1}{I}\left(\int IQ\,dx + C\right) = \frac{1}{x^4}\left(\int x^4 \frac{\sin^3 x}{\cancel{x^4}}\,dx + C\right) = \frac{1}{x^4}\left(\int \sin^3 x\,dx + C\right)$$

$$\int \sin^2 x \sin x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

$$u = \cos x$$

$$du = -\sin x\,dx$$

$$= \int (1-u^2)(-1)\,dx = \frac{1}{3}u^3 - u$$

$$= \frac{1}{3}\cos^3 x - \cos x$$

$$Y = \frac{1}{x^4}\left(\frac{1}{3}\cos^3 x - \cos x + C\right)$$

b. $t^2\frac{dy}{dt}+3ty=\sqrt{1+t^2}, t>0$

$$y' = \frac{\sqrt{1+t^2}}{t^2} - \frac{3}{t}y$$

$$I = e^{\int P\,dt} = e^{\int \frac{3}{t}dt} = e^{3\ln|t|} = |t|^3$$
$$= t^3 \ (t>0)$$

$$y' + \frac{3}{t}y = \frac{\sqrt{1+t^2}}{t^2}$$

$$P(t) = \frac{3}{t} \qquad Q(t) = \frac{\sqrt{1+t^2}}{t^2}$$

$$y = \frac{1}{I}\left(\int I Q\,dt + C\right) = \frac{1}{t^3}\left(\int t^3 \cdot \frac{\sqrt{1+t^2}}{t^2}\,dt + C\right) = \frac{1}{t^3}\left(\int t\sqrt{1+t^2}\,dt + C\right)$$

$$u = 1+t^2$$
$$du = 2t\,dt$$

$$= \frac{1}{t^3}\left(\int \frac{1}{2}\sqrt{u}\,du + C\right)$$

$$= \frac{1}{t^3}\left(\frac{1}{3}u^{3/2} + C\right) = \boxed{\frac{1}{t^3}\left(\frac{1}{3}(1+t^2)^{3/2} + C\right)}$$

$$y x' - 2x = y^2 , \quad y > 0$$

$$x' - \frac{2}{y} x = y$$

$$I = e^{\int P\,dy} = e^{\int \frac{-2}{y} dy} = e^{-2\ln|y|} = |y|^{-2}$$

$$= y^{-2}$$

$$P(y) = \frac{-2}{y} \qquad Q(y) = y$$

$$x = \frac{1}{I}\left(\int I Q\,dy + C\right) = y^2\left(\int y^{-2} y\,dy + C\right)$$

$$= y^2\left(\ln|y| + C\right)$$

$$\boxed{x = y^2\left(\ln(y) + C\right)}$$

$$y' = f(x)y$$

$$y' - f(x)y = 0$$

$$P(x) = -f(x)$$

$$Q(x) = 0$$

$$I = e^{\int -f(x)\,dx} = e^{-\int f(x)\,dx} = \frac{1}{e^{\int f(x)\,dx}}$$

$$y = \frac{1}{I}\left(\int IQ\,dx + C\right)$$

$$= e^{\int f(x)\,dx}\left(\cancel{\int I \cdot 0\,dx} + C\right) = Ce^{\int f(x)\,dx}$$

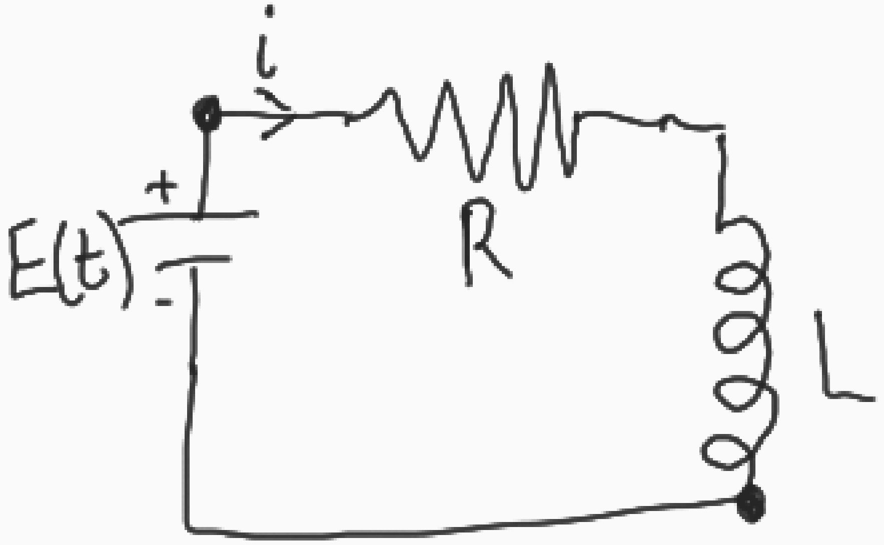
$$\frac{y'}{y} = f(x)$$

$$\int \frac{dy}{y} = \int f(x)\,dx$$

$$\ln|y| = \int f(x)\,dx + C$$

$$y = \pm e^{\int f(x)\,dx + C}$$

$$= Ae^{\int f(x)\,dx}, \quad A = \pm e^C$$



$$E(t) = iR + L\frac{di}{dt}$$

$$\left[\; i' + \frac{R}{L} i = \frac{1}{L}E(t)\right.$$

$$P(t) = \frac{R}{L} \qquad Q(t) = \frac{1}{L}E(t)$$

$$I = e^{\int \frac{R}{L} dt} = e^{Rt/L}$$

$$i = \frac{1}{I}\left(\int I Q\,dt + C\right) = e^{-Rt/L}\left(\int e^{Rt/L}\frac{1}{L}E(t)\,dt + C\right)$$

ex $E(t) = V$,

$$i = e^{-Rt/L}\left(\int e^{Rt/L}\frac{V}{L}dt + C\right)$$

$$= e^{-Rt/L}\left(\frac{V}{R}e^{Rt/L} + C\right) = \frac{V}{R} + Ce^{-Rt/L}$$

$$i_0 = i(0) = \frac{V}{R} + Ce^{-R\cdot 0/L} = \frac{V}{R} + C, \quad C = i_0 - \frac{V}{R}$$

$$i(t) = \frac{V}{R} + \left(i_0 - \frac{V}{R}\right)e^{-Rt/L}$$