9.5 : Linear equations (Separable : $y' = Q(x)P(y)$)

$$y' + P(x)y = Q(x)$$

integrating factor: $I(x) = \exp\left(\int P(x)\,dx\right)$

Then: $y = \frac{1}{I(x)}\left(\int I(x)Q(x)\,dx + C\right)$

$$\begin{aligned}
\text{eqn} \iff I(x)\left(y' + P(x)y\right) &= I(x)Q(x) \\
\iff I(x)y' + I(x)P(x)y &= I(x)Q(x) \\
\iff \left(I(x)y\right)' &= I(x)Q(x) \\
\iff I(x)y &= \int I(x)Q(x)\,dx \\
\iff y &= \frac{1}{I(x)}\int I(x)Q(x)\,dx
\end{aligned}$$

$$\begin{aligned}
I'(x) &= \left(e^{\int P(x)dx}\right)' \\
&= e^{\int P(x)dx}\left(\int P(x)\,dx\right)' \\
&= e^{\int P(x)dx}P(x) \\
&= I(x)P(x)
\end{aligned}$$

$$\begin{aligned}
\left(I(x)y\right)' &= I(x)y' + I'(x)y \\
&= I(x)y' + I(x)P(x)y
\end{aligned}$$

ex $\quad y' + y = 1 \qquad\qquad y' + P(x)y = Q(x)$

$P(x) = 1$
$Q(x) = 1$

$$I(x) = \exp\left(\int P(x)\,dx\right) = \exp\left(\int dx\right) = e^x$$

$$y = \frac{1}{e^x}\left(\int e^x \cdot 1\,dx + C\right) = \frac{1}{e^x}\left(e^x + C\right) = 1 + Ce^{-x}$$

$$y' = -Ce^{-x} \quad \checkmark$$

integrating factor: $I(x) = \exp\left(\int P(x)\,dx\right)$

Then: $y = \frac{1}{I(x)}\left(\int I(x)\,Q(x)\,dx + C\right)$

1. Solve the differential equation:

a. $4x^3y + x^4y' - \sin^3 x$

$$\frac{4}{x}y + y' = \frac{\sin^3 x}{x^4}$$

$$P(x) = \frac{4}{x}$$

$$Q(x) = \frac{\sin^3 x}{x^4}$$

$$I(x) = \exp\left(\int \frac{4}{x}\,dx\right) = \exp\left(4\ln|x|\right) = e^{4\ln|x|} = \left(e^{\ln|x|}\right)^4 = |x|^4 = x^4$$

$$y = \frac{1}{I}\int IQ\,dx = \frac{1}{I}\int x^4\,\frac{\sin^3 x}{x^4}\,dx$$

$$= \frac{1}{x^4}\int \sin^3(x)\,dx$$

$$= \frac{1}{x^4}\int \sin(x)\left(1-\cos^2 x\right)dx$$

$u = \cos x$

$du = -\sin x \; dx$

$$= \frac{1}{x^4}\int -\left(1-u^2\right)du$$

$f(x)^n = \left(f(x)\right)^n$

$$= \frac{1}{x^4}\left(-u + \frac{1}{3}u^3 + C\right)$$

$$= \frac{1}{x^4}\left(-\cos(x) + \frac{1}{3}\cos(x)^3 + C\right)$$

b. $t^2 \frac{dy}{dt} + 3ty = \sqrt{1+t^2}, t > 0$

$$y' + \frac{3y}{t} = \frac{\sqrt{1+t^2}}{t} = \sqrt{t^{-2}+1}$$

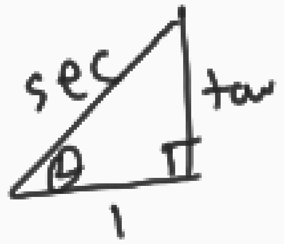
$$I(t) = \exp\left(\int \frac{3}{t}\, dt\right) = \exp(3 \ln t) = \left(e^{\ln(t)}\right)^3 = t^3$$

$$P(t) = \frac{3}{t}$$

$$Q(t) = \sqrt{1+t^{-2}}$$

$$y = \frac{1}{I} \int I Q \, dt = t^{-3} \int t^3 \sqrt{t^{-2}+1}\, dt$$

$$= t^{-3} \int t^2 \sqrt{1+t^2}\, dt$$



$$t = \tan\theta$$

$$dt = \sec^2\theta \, d\theta$$

$$= t^{-3} \int \tan^2\theta \sec\theta \sec^2\theta \, d\theta$$

$$= t^{-3} \int \tan^2\theta \sec^3\theta \, d\theta$$

c. $xy' - 2y = x^2, x > 0$

$$y' - \frac{2}{x}y = x$$

$$P(x) = \frac{-2}{x}$$

$$Q(x) = x$$

$$I(x) = \exp\int -\frac{2}{x}\,dx = \exp\left(-2\ln|x|\right) = |x|^{-2} = x^{-2}$$

$$y = \frac{1}{I}\int IQ\,dx = x^2\int x^{-2}x\,dx$$

$$= x^2\left(\ln|x| + C\right)$$

$$= x^2\left(\ln(x) + C\right)$$

$$y' = x^2\left(\frac{1}{x} + 0\right) + 2x\left(\ln(x) + C\right)$$

$$= x + 2x\ln(x) + 2xC \quad \checkmark$$

$$E(t) = iR + L\frac{di}{dt}$$

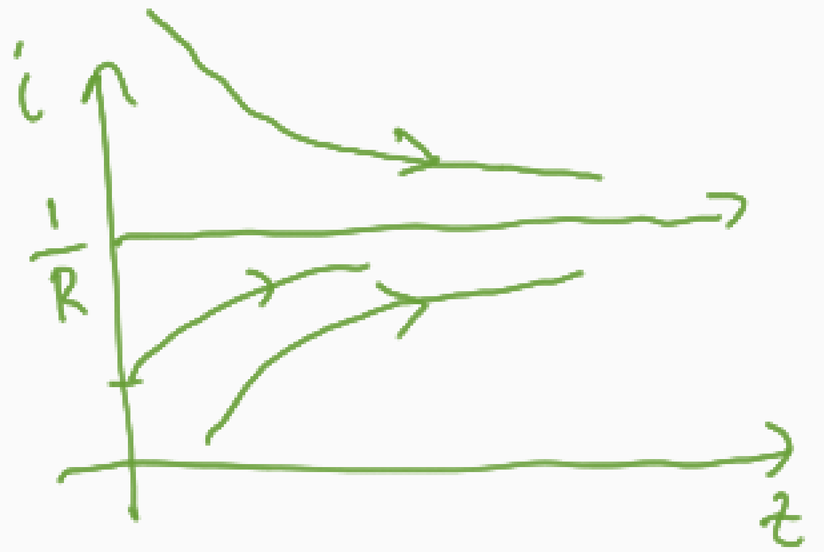
$$i' + \frac{R}{L}i = \frac{1}{L}E(t)$$

$$P(t) = \frac{R}{L}$$

$$Q(t) = \frac{1}{L}E(t)$$

$$i_0 = \frac{1}{R} + Ce^{-R\cdot 0/L} = \frac{1}{R} + C$$

$$i_0 - \frac{1}{R} = C$$



$$I(t) = \exp\left(\int \frac{R}{L}\,dt\right) = e^{Rt/L}$$

$$i(t) = e^{-Rt/L}\left(\int e^{Rt/L}\frac{1}{L}E(t)\,dt + C\right)$$

ex $E(t) = 1$

$$i(t) = e^{-Rt/L}\left(\int e^{Rt/L}\frac{1}{L}\,dt + C\right) = e^{-Rt/L}\left(\frac{1}{R}e^{Rt/L} + C\right)$$

$$= \frac{1}{R} + Ce^{-Rt/L}$$

$$i(0) = i_0 \rightsquigarrow i(t) = \frac{1}{R} + \left(i_0 - \frac{1}{R}\right)e^{-Rt/L}$$