Some integration by parts

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In the 3pm discussion section, we nobly struggled through one of the rather tricky integrals from the lecture worksheet. While we managed to find the antiderivative in the end, what we did was mostly scratchwork. It's important to do the next step which is to try to refine the solution and write up a clean answer.

The integral in question is $\int \frac{1}{(1+x^2)^2}\,dx$.

1. We introduce $x^2 - x^2$ in the numerator of the integrand:

$$\frac{1}{(1+x^2)^2} = \frac{1+x^2-x^2}{(1+x^2)^2} = \frac{1+x^2}{(1+x^2)^2} - \frac{x^2}{(1+x^2)^2} = \frac{1}{1+x^2} - \frac{x^2}{(1+x^2)^2}.$$

2. Hence,

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx - \int \frac{x^2}{(1+x^2)^2}\,dx = \tan^{-1}(x) - \int \frac{x^2}{(1+x^2)^2}\,dx.$$

3. (a) Subproblem: $\int \frac{x^2}{(1+x^2)^2}\,dx$.

   (b) We will use integration by parts with $f(x) = x$ and $g'(x) = \frac{x}{(1+x^2)^2}$.

   (c) i. Subproblem: $\int \frac{x}{(1+x^2)^2}\,dx$.

   ii. Consider the u-substitution $u = 1+x^2$, where $du = 2x\,dx$ (so $x\,dx = \frac{1}{2}\,du$).

   iii. Then

$$\int \frac{x}{(1+x^2)^2}\,dx = \int \frac{1}{2u^2}\,du = -\frac{1}{2u} = -\frac{1}{2(1+x^2)}$$

   (plus a constant).

   (d) Now we can fill out the integration by parts table:

$$f(x) = x \qquad g'(x) = \frac{x}{(1+x^2)^2}$$

$$f'(x) = 1 \qquad g(x) = -\frac{1}{2(1+x^2)}.$$

   (e) Integration by parts gives

$$\begin{aligned}\int \frac{x}{(1+x^2)^2}\,dx &= x\frac{-1}{2(1+x^2)} - \int 1\cdot\frac{-1}{2(1+x^2)}\,dx \\ &= -\frac{x}{2(1+x^2)} + \frac{1}{2}\int \frac{1}{1+x^2}\,dx = -\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}(x) + C.\end{aligned}$$

4. Substituting in the solved subproblem,

$$\begin{aligned}\int \frac{1}{(1+x^2)^2}\,dx &= \tan^{-1}(x) - \left(-\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}(x) + C\right) \\ &= \frac{1}{2}\tan^{-1}(x) + \frac{x}{2(1+x^2)} - C.\end{aligned}$$

That's as clean as I personally could make it. And remember: this was the result of lots of trial and error! There is pretty much no way anyone would be able to write this solution straight-through on a first try, barring having lots of hard-won experience. (I think about it like we wandered through a maze, found the exit, then wrote down optimized directions.)

One takeaway is that sometimes the $g'(x)$ you choose might take some effort to integrate, so don't overlook that possibility in your own efforts.

1 Trigonometric substitution

I made a comment that "by parts is inappropriate for this problem" because there is a technique that we'll be learning later that will make it much easier. (I should say that integration by parts *does* work, so don't take me too seriously about inappropriateness.) Just for illustration, I'll give you a preview.

1. Recall $\sin(\theta)^2 + \cos(\theta)^2 = 1$, hence $\tan(\theta)^2 + 1 = \sec(\theta)^2$.
2. Perform the substitution $x = \tan(\theta)$ (so then $dx = \sec(\theta)^2\, d\theta$):
$$\int \frac{1}{(1+x^2)^2}\, dx = \int \frac{1}{(\sec(\theta)^2)^2} \sec(\theta)^2\, d\theta = \int \cos(\theta)^2\, d\theta.$$
3. There are a few ways to deal with this integral (try doing it by parts!), but I'm going to use the half angle formula $\cos(\theta)^2 = \frac{1}{2}(1+\cos(2\theta))$:
$$= \int \frac{1}{2}(1+\cos(2\theta))\, d\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C$$
4. Substituting $\theta = \tan^{-1}(x)$, this is
$$= \frac{1}{2}\left(\tan^{-1}(x) + \frac{1}{2}\sin(2\tan^{-1}(x))\right) + C$$
5. Using trig identities, you can get that $\sin(2\tan^{-1}(x)) = \frac{x}{1+x^2}$.

2 Inappropriate integration by parts

I think this is a kind of funny use of integration by parts: $\int x^2\, dx$.

1. We will do integration by parts with
$$f(x) = x \qquad g'(x) = x$$
$$f'(x) = 1 \qquad g(x) = \frac{1}{2}x^2.$$
2. Then
$$\begin{aligned}\int x^2\, dx &= x \cdot \frac{1}{2}x^2 - \int 1 \cdot \frac{1}{2}x^2 \\ &= \frac{1}{2}x^3 - \frac{1}{2}\int x^2\, dx.\end{aligned}$$
3. Adding $\frac{1}{2}\int x^2\, dx$ to both sides, this gives $\frac{3}{2}\int x^2\, dx = \frac{1}{2}x^3$, hence we get the antiderivative
$$\int x^2\, dx = \frac{1}{3}x^3 + C.$$