Terminal velocity of a water droplet

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Today there was an interesting problem in the book which did not require solving differential equations to analyze. Here it is, paraphrased:

> Water droplets gain mass as they fall (presumably by collecting water vapor along their path). Pretending the cross-sectional area of a water droplet is in proportion to its mass, we model this as $m' = km$ for some positive constant k. Newton's law of motion is that force is mass times acceleration, and since the derivative of momentum is force, we get $(mv)' = mg$, where v is the downward velocity of a raindrop, and g is acceleration due to gravity. Find an expression for the terminal velocity of the raindrop, where terminal velocity is defined to be $\lim_{t\to\infty} v(t)$.

One thing to notice about this problem is that m is a function of time, which is quite the departure from high-school physics. Thinking about energy, if an object is moving without any force acting on it, and it starts to gain mass, then its velocity must decrease for the energy to remain constant; this is basically why the raindrop will achieve some terminal velocity rather than get faster and faster without bound. (An aside: a better model would include drag and would have a non-exponential droplet growth model, though the resulting differential equation would probably be difficult to solve.)

The first thing to note is that $(mv)'$ is the derivative of a product where both m and v are functions of time, so the correct derivative is $m'v+mv'$. Since $m' = km$, we may substitute it to get $m(kv+v')$. Dividing by m on both sides, we have the differential equation $kv+v' = g$. (Notice this is independent of mass — I am certain Stewart chose the growth model for the drops so that this would happen.)

Rewriting the equation, we have $v' = k(\frac{g}{k} - v)$. In this form, we see that an equilibrium solution is $v = \frac{g}{k}$. In fact, the solution is stable because when $v > \frac{g}{k}$ we have $v' < 0$ and when $v < \frac{g}{k}$ we have $v' > 0$. With a lemma (below), if $v(t)$ is a solution to the differential equation, then $\lim_{t\to\infty} v(t) = \frac{g}{k}$. One could also solve the differential equation and compute the limit, but I believe it is more enlightening to see that the differential equation is directly causing functions to converge to the equilibrium. (For completeness' sake, one should note that the equation has at least one solution. In particular, the equilibrium.)

The following lemma is a detail for which I faltered giving a quick explanation. Basically, we had no way of knowing *a priori* that, for a general differential equation with a single equilibrium solution which is stable, that every solution must have $\lim_{t\to\infty} v'(t) = 0$, so why should we believe the equilibrium solution *is* the terminal velocity? A non-rigorous answer is that when a function is less than the equilibrium then it is increasing, when it is greater then it is decreasing, and when it is at the equilibrium it is neither increasing nor decreasing, so the limit *must* be the equilibrium value. However, the lemma will answer this rigorously:[1]

Lemma 1. *Consider a differential equation $\frac{dy}{dx} = F(y)$ with F a continuous function and c a number where $F(c) = 0$. If $F(y)$ is positive when $y < c$ and negative when $y > c$ then for any solution f to the differential equation, $\lim_{x\to\infty} f(x) = c$.*

That is, if c is the only equilibrium for some differential equation, and if it is a stable equilibrium, then every solution to the differential equation will approach c at infinity.

Proof. Let f be a solution to the differential equation, and let c be a number where $F(c) = 0$. We would like to show that $\lim_{x\to\infty} f(x) = c$, and to do this we must consider a few things which could happen. (A key idea will be to show that a solution which never actually equals the equilibrium indeed has $\lim_{x\to\infty} f'(x) = 0$, and to use continuity of F to show that then f must approach c.)

1. (What if f ever achieves the equilibrium?) Suppose a is some number for which $f(a) = c$. We will show that then f must thereafter be constant c. So for sake of contradiction, suppose there is some $x > a$ where $f(x) \neq c$.

   (a) There is some number β between a and x (possibly equal to a) where $f(\beta) = c$ and where f is never again equal to c between β and x. (That is, there is point where, on the interval $[a, x]$, f equals c for the last time.):

      i. Consider the set of all $\alpha \in [a, x)$ such that $f(\alpha) = c$. There is at least one element in this set (a in particular), and the set has x as an upper bound. Then by the completeness axiom for the real numbers, there is a least upper bound for this set, call it β. It must be that $f(\beta) = c$ and that $\beta < x$:

         A. There is a sequence of numbers (not necessarily distinct) $\alpha_1, \alpha_2, \ldots$ such that $f(\alpha_i) = c$ and such that $\lim_{i\to\infty} \alpha_i = \beta$. (One way to do this: for each i, take some number from the set [and call it α_i] which is within a distance of $\frac{1}{i}$ of β. There is always such an α_i because otherwise β wouldn't be a least upper bound.)

         B. Since f is continuous, $f(\beta) = f(\lim_{i\to\infty} \alpha_i) = \lim_{i\to\infty} f(\alpha_i) = \lim_{i\to\infty} c = c$.

         C. We know $\beta < x$, since otherwise $c = f(\beta) = f(x) \neq c$, a contradiction!

   (b) By the Mean Value Theorem, there is some γ between β and x such that

   $$\frac{f(x) - f(\beta)}{x - \beta} = f'(\gamma).$$

   (c) If $f(x) > c$, then since $f(\beta) < f(x)$, $f'(\gamma)$ is positive, so $0 < f'(\gamma) = F(f(\gamma))$. Since F is only positive when its argument is less than c, $f(\gamma) < c$. By the Intermediate Value Theorem, since $f(\gamma) < c$ and $f(x) > c$, there is some ξ between γ and x such that $f(\xi) = c$. However, $\xi > \gamma > \beta$, which contradicts β being the largest number smaller than x where f equals c.

[1] All of the techniques in this proof are material from 1A and 1B. While you are not expected to be able to come up with such a proof entirely on your own, ideally each individual step is within reach.

(d) If $f(x) < c$, then a similar argument can be made to get a contradiction.

Thus, for all $x > a$, $f(x) = c$. That is, once f achieves the equilibrium value, it never leaves equilibrium.

2. (What if f neither entirely above nor entirely below the equilibrium value?) If f is less than c somewhere and greater than c somewhere else, by the Intermediate Value Theorem it is exactly c somewhere between, but then we have a contradiction: we just showed that if it ever is exactly c, it will be c forevermore.

3. (What if f is entirely below the equilibrium value?) Now we consider the case where $f(x) < c$ for all x. Since then $f'(x) = F(f(x)) > 0$, we have $f(x)$ being an increasing function. By the completeness axiom for the real numbers, since $f(x) < c$ for all x, then $\lim_{x\to\infty} f(x) = \alpha$ for some α (the limit exists).

   By continuity we have $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} F(f(x)) = F(\lim_{x\to\infty} f(x)) = F(\alpha)$. We will show $F(\alpha) = 0$, and therefore, since c is the only place where F is zero, that $\alpha = c$.

   Suppose $\lim_{x\to\infty} f'(x) = \beta > 0$, for sake of contradiction.

   (a) Then by the definition of the limit, there is some M and some $k > 0$ such that for all $x > M$, $f'(x) > k$.

   (b) By the Mean Value Theorem, for each $x > M$, there is some γ between M and x such that
   $$\frac{f(x) - f(M)}{x - M} = f'(\gamma),$$

   (c) and since $\gamma > M$, then $f'(\gamma) > k$.

   (d) So, $f(x) - f(M) > k(x - M)$ which implies $f(x) > f(M) + k(x - M)$.

   (e) However, this implies that if $x > M + \frac{c - f(M)}{k}$ (which definitely will eventually happen) then $f(x) > c$, which contradicts $f(x)$ being always less than c.

   Thus, $\lim_{x\to\infty} f'(x) = 0$, so $F(\alpha) = 0$, so $\alpha = c$. Therefore, $\lim_{x\to\infty} f(x) = c$.

4. (What if f is entirely above the equilibrium value?) The case where $f(x) > c$ for all x is in essence the same argument.

Every solution to the differential equation will fall into at least one of these cases, so therefore $\lim_{x\to\infty} f(x) = c$ for any solution f to the differential equation $\frac{dy}{dx} = F(y)$ when c is the only equilibrium solution to F, and when c is a stable equilibrium. $\square$