Hydrostatic pressure in a cylindrical tank

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There was a problem on the homework due today which I did correctly in the 10am section, then got confused about how pressure works (a confusion which stayed with me through the noon section).

The problem was this: given a horizontal cylindrical tank of radius r which is full of liquid, what is the force on one of its circular faces? To solve this, all we need to do is integrate pressure over the entire circular face. However, what is the pressure at a point of the face? There are two real options: 1) the pressure is in proportion to the depth relative to the highest liquid in the container, or 2) the pressure is in proportion to the the distance to the piece of container immediately above it.

It is not the second one. Consider the following setup (figure 1): At some arbitrary point along the circular face, we cut a very small hole and attach a very narrow vertical tube. Fluid will flow into the tube until it is at the same height as the top of the tank. Interestingly, this acts as a rudimentary pressure gauge, since the pressure on either side of the tap must be equal, and the pressure on the tube side is in proportion to the height of the column of fluid in it.

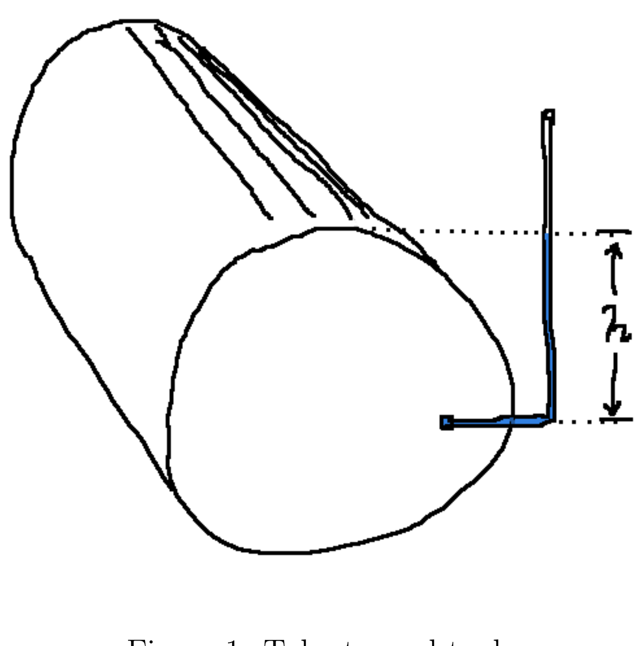

Figure 1: Tube-tapped tank.

Thus, the pressure at any point on the circular face really is just $\rho g h$. With coordinates (x, y) relative to the center, $h = r - y$, and so the force is the following integral:

$$\begin{aligned}
F &= \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \rho g(r - y)\, dx\, dy \\
&= \int_{-r}^{r} 2\rho g(r - y)\sqrt{r^2 - y^2}\, dy \\
&= 2\rho g \left(r \int_{-r}^{r} \sqrt{r^2 - y^2}\, dy - \int_{-r}^{r} y\sqrt{r^2 - y^2}\, dy \right) \\
&= 2\rho g (r \frac{\pi r^2}{2} - 0) \\
&= \pi \rho g r^3,
\end{aligned}$$

where the first integral is the area of a semicircle of radius r, and the second has an integrand which is an odd function.